ELEMENTARY PARTIAL DIFFERENTIAL EQUATIONS: CHARACTERISTICS, CONSERVATION, AND TRAFFIC WAVES

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1. Introduction

A partial differential equation (PDE) is an equation that relates certain partial derivatives of a function. A PDE, together with additional conditions (such as initial conditions or boundary conditions), often determines the function uniquely, just as Newton's law (an ordinary differential equation), together with initial conditions, determines the time evolution of a mechanical system.

PDEs are fundamental in many fields of science. Examples include: the minimal surface equation in geometry; the Schrödinger equation in quantum mechanics; the Navier-Stokes equation of fluid dynamics; the reaction-diffusion equation governing electrochemical waves in heart muscle; and the Black-Scholes equation for pricing stock options.

In these notes, we discuss some of the simplest PDEs, ones that can be solved using the so-called Method of Characteristics. We will use the *Maple* program to help us construct, visualize, and explore solutions. As an application, we will study some wave phenomena occurring in a model of traffic dynamics.

2. Convection

Let us start with the simplest initial-value problem. We are given a function u_0 of $x \in \mathbb{R}$ (the *initial data*), and we are to find a function u of the variables $x \in \mathbb{R}$ and $t \geq 0$ such that

$$\frac{\partial u}{\partial t}(x,t) = 0 \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{2.1}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}. \tag{2.2}$$

In this example, Eq. (2.1) is the *partial differential equation* for the unknown function u, and Eq. (2.2), which specifies the values for u at the initial time $t = 0$, is the *initial condition.*

What does Eq. (2.1) say? It says simply that, for each fixed x, $u(x,t)$ does not change with t. Equation (2.2) specifies the value of $u(x,t)$ at a particular t, namely $t = 0$, and the value of $u(x,t)$ for all $t > 0$ must be the same. Therefore the solution is

$$u(x,t) = u_0(x) \quad \text{for } x \in \mathbb{R} \text{ and } t \geq 0. \tag{2.3}$$

Of course, this example is trivial. Nonetheless, it will guide us in solving more difficult PDEs. For instance, consider the following initial-value problem:

$$\frac{\partial u}{\partial t}(x,t) + c\frac{\partial u}{\partial x}(x,t) = 0 \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{2.4}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}. \tag{2.5}$$

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Equation (2.4) is called a *convection* (or *advection* or *transport*) equation; c is a constant that, as we will see, is the velocity of convection.

To solve this initial-value problem we use the following procedure. Introduce a new independent variable $\xi \in \mathbb{R}$ and express the independent variable x as

$$x = X(\xi, t) = \xi + ct. \tag{2.6}$$

Under this correspondence, let $U(\xi, t) = u(x, t)$; in other words, define U by

$$U(\xi, t) = u(X(\xi, t), t) = u(\xi + ct, t). \tag{2.7}$$

Now notice that, by the chain rule,

$$\begin{aligned}\frac{\partial U}{\partial t}(\xi, t) &= \frac{\partial u}{\partial t}(X(\xi, t), t) + \frac{\partial u}{\partial x}(X(\xi, t), t)\,\frac{\partial X}{\partial t}(\xi, t)\\ &= \frac{\partial u}{\partial t}(\xi + ct, t) + \frac{\partial u}{\partial x}(\xi + ct, t)\,c\\ &= 0.\end{aligned}$$

Also, $U(\xi, 0) = u(\xi, 0) = u_0(\xi)$. Therefore the initial-value problem for u translates into the initial-value problem

$$\frac{\partial U}{\partial t}(\xi, t) = 0 \qquad \text{for } \xi \in \mathbb{R} \text{ and } t > 0, \tag{2.8}$$

$$U(\xi, 0) = u_0(\xi) \quad \text{for } \xi \in \mathbb{R} \tag{2.9}$$

for U. Thus U satisfies the simplest initial-value problem (2.1)–(2.2), which entails that

$$U(\xi, t) = u_0(\xi) \quad \text{for } \xi \in \mathbb{R} \text{ and } t > 0. \tag{2.10}$$

As

$$\xi = x - ct, \tag{2.11}$$

we find that $u(x, t) = U(x - ct, t)$, so that

$$u(x, t) = u_0(x - ct) \quad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{2.12}$$

Example 2.1. The initial-value problem

$$\frac{\partial u}{\partial t}(x, t) + 2\frac{\partial u}{\partial x}(x, t) = 0 \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{2.13}$$

$$u(x, 0) = \exp(-x^2) \quad \text{for } x \in \mathbb{R}. \tag{2.14}$$

has solution

$$u(x, t) = \exp(-(x - 2t)^2). \tag{2.15}$$

Exercise 2.2. Verify this.

The only seemingly *ad hoc* step in the procedure described above is the change of variables (2.6). In fact, for each fixed $\xi \in \mathbb{R}$, the map $t \mapsto X(\xi, t)$ is the solution of the ordinary differential equation initial-value problem

$$\frac{dx}{dt}(t) = c \quad \text{for } t > 0, \tag{2.16}$$

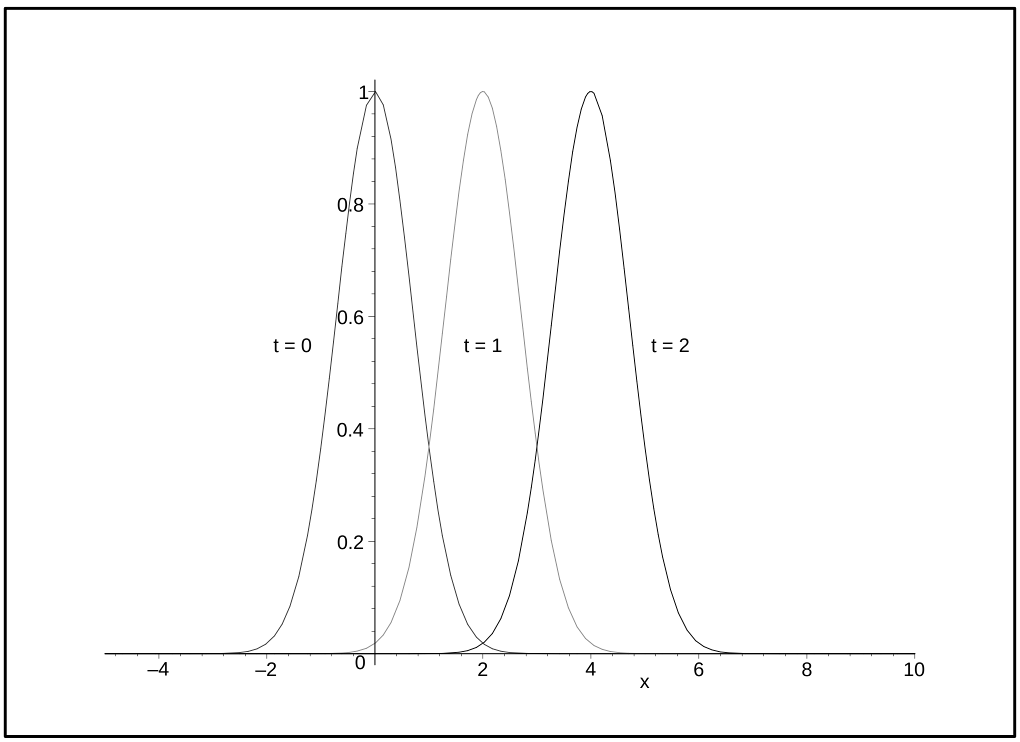
$$x(0) = \xi. \tag{2.17}$$

We will see shortly how to generalize this observation.

3. Visualization

Let us consider various ways in which we can visualize the solution (2.12) the initial-value problem (2.4)–(2.5). For definiteness, in the following plots, let us take $c = 2$ and $u_0(x) = \exp(-x^2)$. First, we can plot u *vs.* x for various choices of t, say $t = 0$, $t = 1$, and $t = 2$.

```
> with(plots):
> c := 2:
  u_0 := x -> exp(-x^2):
  u := (x,t) -> u_0(x - c*t):
> p1 := plot(u(x,0), x=-5..10, color=red):
  p2 := plot(u(x,1), x=-5..10, color=green):
  p3 := plot(u(x,2), x=-5..10, color=blue):
> t0 := textplot([-1.5, .55, "t = 0"]):
  t1 := textplot([2, .55, "t = 1"]):
  t2 := textplot([5.5, .55, "t = 2"]):
> display({p1,p2,p3,t0,t1,t2});
```



Evidently, the plot of u *vs.* x at time $t > 0$ is the same as the plot of the initial data except for a shift by a distance ct. In other words, the initial profile simply moves, or convects, at velocity c. The profile moves toward the right when $c > 0$, and toward the left when $c < 0$.

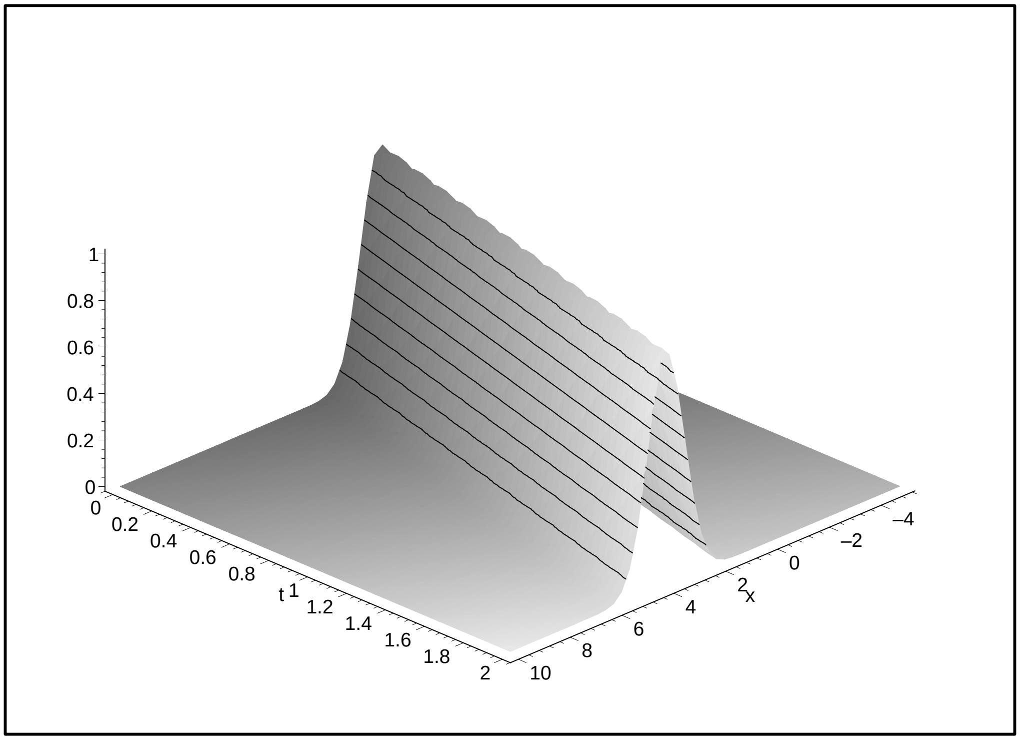
Maple allows us to get a sense of the motion of the solution through its animation facilities. Try the following command.

```
> animate(u(x,t), x=-5..10, t=0..2, numpoints=100, frames=30);
```

If the left mouse button is clicked over the plot, the toolbar at the top of the *Maple* window shows buttons for manipulating the animation. For instance, the button with the large right-pointing arrow starts the animation. In the `animate` command, `numpoints` is the number of points along the x-axis used to plot the function. The default value of 50 points is not sufficient—the tip of the moving profile is poorly resolved. (Try it!) Also, `frames` is the number of values of t for which the profile is plotted; its default value is 16.

Another way to visualize the solution is with a three-dimensional plot of u *vs.* x and t. The solutions $u(x,t)$ define a two-dimensional surface, parameterized by x and t, which is drawn in (x,t,u)-space:

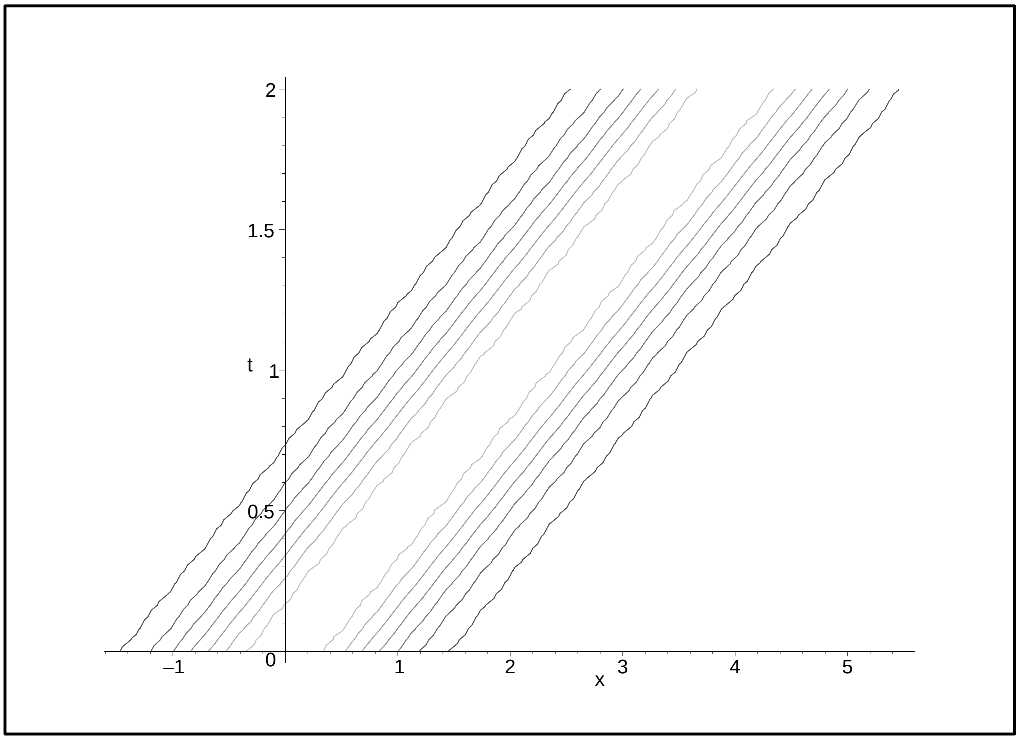
```
> plot3d(u(x,t), x=-5..10, t=0..2, grid=[50,50], style=patchcontour,
  axes=framed);
```



To get better resolution of the surface, we have chosen the number of grid points in both the x and the t directions to be 50 (the default value being 25). Also, the `style` option is set to `patchcontour`, which means that some level curves of the surface are drawn. (See the discussion below about contour plots.) The direction from which the plot is viewed can be controlled: dragging the cursor with the left mouse button depressed shows a wireframe cube that is rotated by the movement of the mouse; pressing the `Enter` key redraws the plot. There are many other options for controlling the appearance of the plot, and some of them are available on a menu that pops up when the right mouse button is clicked while the cursor is over the plot.

Finally, we can visualize the solution using a contour plot. A contour is a curve in the (x,t)-plane along which $u(x,t) = k$, where k is constant, and a contour plot comprises contours corresponding to several values of k. For the solution of the convection initial-value problem, *Maple* draws the following contour plot.

```
> contourplot(u(x,t), x=-5..10, t=0..2, grid=[50,50]);
```



Differing grayscale intensities (or colors) indicate different values of k.

Each contour consists of two parallel straight lines, placed symmetrically with respect to the trajectory of the peak. Indeed $u(x,t) = u_0(x-ct)$ is constant when $\xi = x - ct$ is constant, so the contours are formed from the lines

$$t = \frac{1}{c}x - \frac{1}{c}\xi \tag{3.1}$$

with ξ fixed. Notice that x is the horizontal axis and t is the vertical axis. Therefore the slope of a contour has units of time over distance; in other words, the *inverse* of the slope has units of velocity. In the present example, the inverse slope is the convection velocity c, namely 2. If c were zero (as it is for Eq. (2.1)), the contours would be vertical lines; if c were negative, the contours would tilt to the left, indicating motion to the left.

4. Method of Characteristics

The procedure used in Sec. 2 to solve the initial-value problem (2.4)–(2.5) has a generalization. Consider, for instance, the initial-value problem

$$\frac{\partial u}{\partial t}(x,t) + c(x,t)\frac{\partial u}{\partial x}(x,t) = 0 \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{4.1}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}. \tag{4.2}$$

Here c is a known functions of x and t; we will find that it still represents the velocity of convection. What has changed in going from Eq. (2.4) to Eq. (4.1) is that the coefficient c depends on x and t. Nevertheless, we can proceed much as before.

For each fixed $\xi \in \mathbb{R}$, let $t \mapsto X(\xi, t)$ be the solution of the ordinary differential equation initial-value problem

$$\frac{dx}{dt}(t) = c(x,t) \quad \text{for } t > 0, \tag{4.3}$$

$$x(0) = \xi. \tag{4.4}$$

The map $t \mapsto (X(\xi,t), t)$ is a parameterized curve in the (x,t)-plane called a *characteristic curve.* Although the initial-value problem (4.3)–(4.4) might be difficult or impossible to solve analytically, let us assume that we have managed to find a solution. (A basic result in the theory of ordinary differential equations guarantees that a solution exists, at least for t in a nonempty interval $[0, T(\xi))$, provided that c is suitably well-behaved.)

Example 4.1. Consider the initial-value problem

$$\frac{\partial u}{\partial t}(x,t) + x\frac{\partial u}{\partial x}(x,t) = 0 \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{4.5}$$

$$u(x,0) = \exp(-(x-4)^2) \quad \text{for } x \in \mathbb{R}. \tag{4.6}$$

Here $c(x,t) = x$: the convection velocity is negative for $x < 0$, is positive for $x > 0$, and increases from left to right. The initial-value problem for a characteristic curve is

$$\frac{dx}{dt}(t) = x \quad \text{for } t > 0, \tag{4.7}$$

$$x(0) = \xi. \tag{4.8}$$

Exercise 4.2. Verify that the solution is

$$x(t) = \xi \exp(t). \tag{4.9}$$

Now imagine that we already had a solution u of the initial-value problem (4.1)–(4.2). Then we could define U by

$$U(\xi,t) = u(X(\xi,t),t) \tag{4.10}$$

and find, using the chain rule, that

$$\begin{aligned} \frac{\partial U}{\partial t}(\xi,t) &= \frac{\partial u}{\partial t}(X(\xi,t),t) + \frac{\partial u}{\partial x}(X(\xi,t),t)\,\frac{\partial X}{\partial t}(\xi,t) \\ &= \frac{\partial u}{\partial t}(X(\xi,t),t) + \frac{\partial u}{\partial x}(X(\xi,t),t)\,c(X(\xi,t),t) \\ &= 0. \end{aligned}$$

Thus U would be constant in t for fixed each $\xi \in \mathbb{R}$. Also, since $\xi = X(\xi, 0)$, we would find that $U(\xi,0) = u(X(\xi,0),0) = u(\xi,0) = u_0(\xi)$, Therefore U would be given by $U(\xi,t) = u_0(\xi)$. In terms of the hypothesized solution u, this means that $u(X(\xi,t),t) = u_0(\xi)$.

The important conclusion is that the PDE (4.1) requires u to be constant along characteristic curves and the initial condition (4.2) specifies what that constant value is.

This calculation suggests that the solution $u(x,t)$ can be obtained by eliminating ξ between the two equations

$$x = X(\xi,t), \tag{4.11}$$

$$u = u_0(\xi). \tag{4.12}$$

To do this, we would solve the equation

$$x = X(\xi, t) \tag{4.13}$$

for ξ as a function of x and t, say $\xi = \Xi(x,t)$, and then set

$$u(x,t) = u_0(\Xi(x,t)). \tag{4.14}$$

Equation (4.13) can be solved, at least for (x,t) in a neighborhood of a point $(\xi, 0)$, by virtue of the Implicit Function Theorem, since $(\partial X/\partial\xi)(\xi,0) = 1$ is nonzero.

Example 4.3. For the initial-value problem of the preceding example, we would solve Eq. (4.9) to find that

$$\xi = x\exp(-t), \tag{4.15}$$

which is the equation $\xi = \Xi(x,t)$. Substituting into the initial condition gives

$$u(x,t) = \exp(-(x\exp(-t) - 4)^2). \tag{4.16}$$

Exercise 4.4. Verify this solution.

In general, the function u defined this way is, indeed, a solution of the initial-value problem (4.1)–(4.2). To see this, first notice that implicit differentiation of Eq. (4.14) shows that

$$\frac{\partial u}{\partial x}(x,t) = u_0'(\Xi(x,t))\frac{\partial \Xi}{\partial x}(x,t), \tag{4.17}$$

$$\frac{\partial u}{\partial t}(x,t) = u_0'(\Xi(x,t))\frac{\partial \Xi}{\partial t}(x,t). \tag{4.18}$$

(Here u_0' denotes the derivative of u_0 with respect to its argument.) Moreover, differentiating the identity $\xi = \Xi(X(\xi,t),t)$ with respect to t shows that

$$\begin{aligned} 0 &= \frac{\partial \Xi}{\partial x}(X(\xi,t),t)\frac{\partial X}{\partial t}(\xi,t) + \frac{\partial \Xi}{\partial t}(X(\xi,t),t) \\ &= \frac{\partial \Xi}{\partial x}(X(\xi,t),t)c(X(\xi,t),t) + \frac{\partial \Xi}{\partial t}(X(\xi,t),t). \end{aligned} \tag{4.19}$$

Taking $\xi = \Xi(x,t)$, we see that

$$0 = \frac{\partial \Xi}{\partial x}(x,t)c(x,t) + \frac{\partial \Xi}{\partial t}(x,t). \tag{4.20}$$

Consequently, multiplying Eq. (4.17) by $c(x,t)$ and adding Eq. (4.18) gives Eq. (4.1). Also, $\Xi(x,0) = x$ because $X(\xi,0) = \xi$, so that $u(x,0) = U(\Xi(x,0),0) = U(x,0) = u_0(x)$, which is Eq. (4.2).

The foregoing method of obtaining a solution is called the *Method of Characteristics.*

5. Using *Maple* to solve PDEs

Maple has a facility, `PDEtools`, that uses the Method of Characteristics to solve initial-value problems for a class of PDEs (namely, first-order scalar PDEs in several variables) that includes the kind we have studied.

To see how `PDEtools` works, first consider the example of Sec. 2. Type the following *Maple* commands.

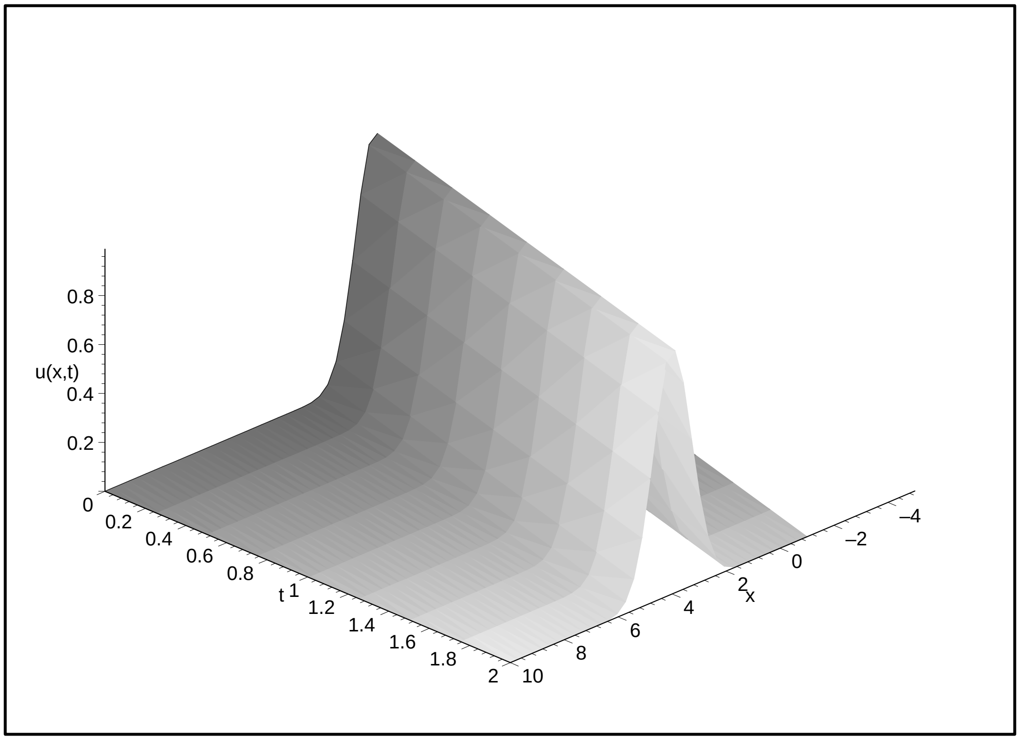
```
> with(PDEtools):
```

```
> convect := diff(u(x,t),t) + 2*diff(u(x,t),x) = 0:
  ic := [s, 0, exp(-s^2)]:
  srange := s=-5..10:
  xrange := x=-5..10:
  trange := t=0..2:
```

The variable `convect` contains the specification of the PDE. The variable `ic` represents the initial condition as a curve in (x, t, u)-space parameterized by the variable s; in the present example, the initial condition $u(x, 0) = \exp(-x^2)$ is represented by $x = s$, $t = 0$, and $u = \exp(-s^2)$ as functions of s. The variables `srange`, `xrange`, and `trange` are ranges for s, x, and t, respectively.

Now type the following command.

```
> PDEplot(convect, ic, srange, xrange, trange, animate=true,
  numchar=50, numsteps=[0,20]);
```



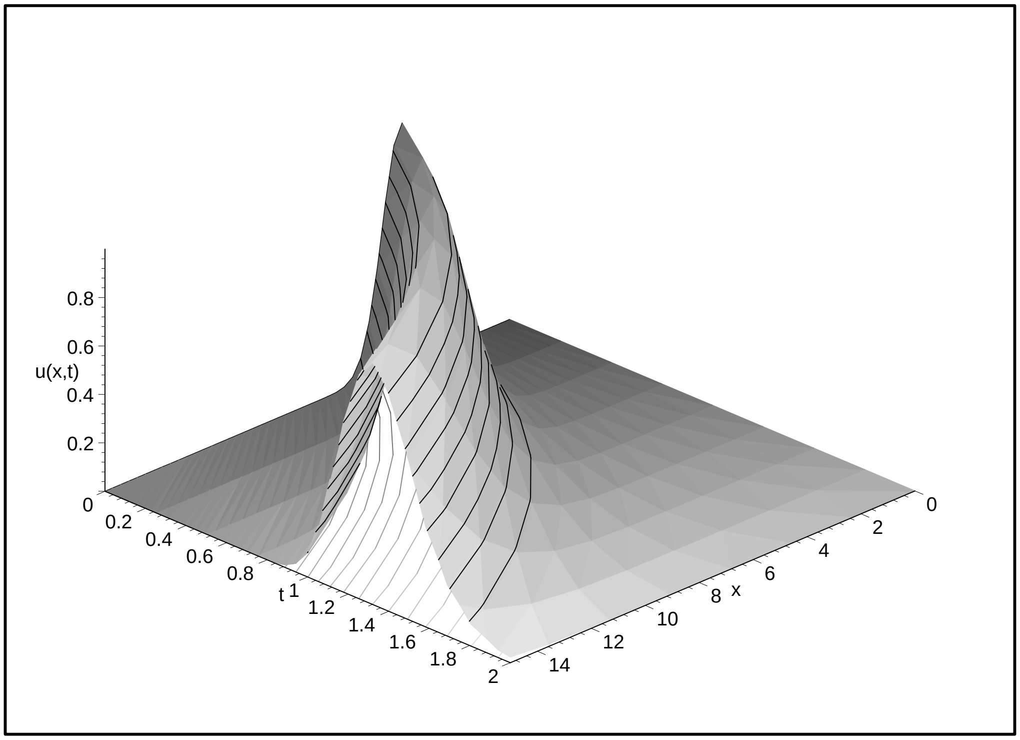
The result is a depiction of the solution as a surface in (x, t, u)-space. Having specified the option `animate=true`, we can also view an animation of the graph of $x \mapsto u(x, t)$ as t varies. (Try it.)

Next consider the example of Sec. 4. This initial-value problem is specified as follows.

```
> convect2 := diff(u(x,t),t) + x*diff(u(x,t),x) = 0:
  ic := [s, 0, exp(-(s-4)^2)]:
  srange := s=0..15:
  xrange := x=0..15:
  trange := t=0..2:
```

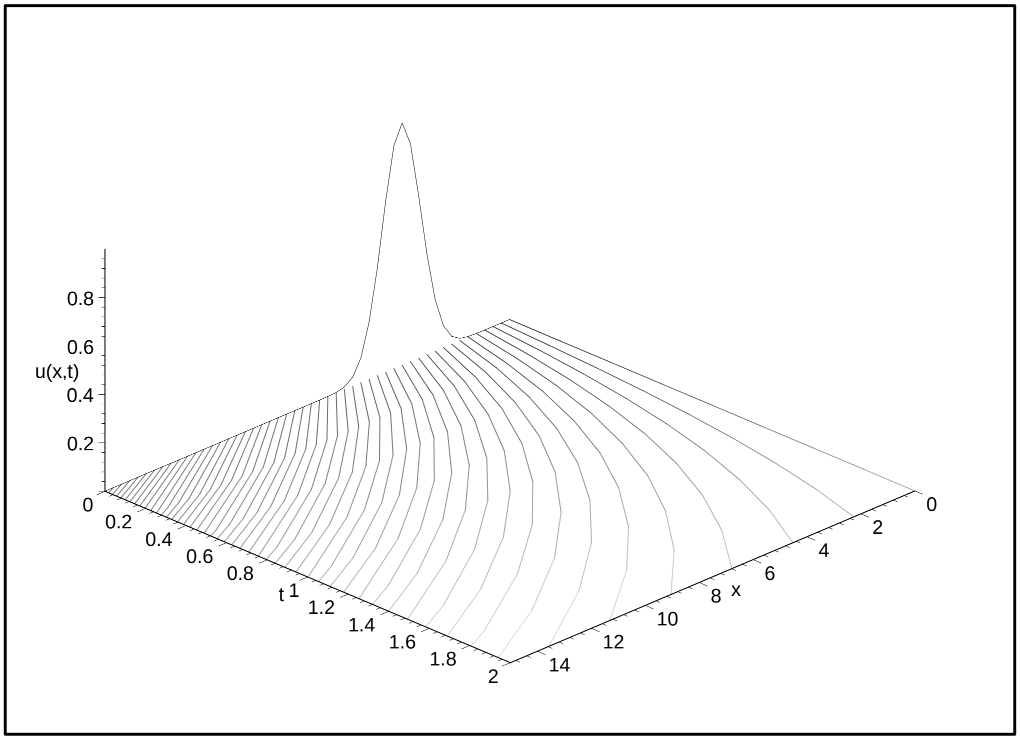
The solution is obtained as follows.

```
> PDEplot(convect2, ic, srange, xrange, trange, animate=true,
  basechar=true, style=patchcontour, numchar=50, numsteps=[0,20]);
```



We have included the `basechar=true` option so that the characteristic curves are drawn. We can also view the characteristic curves alone by using the `basechar=only` option:

```
>  PDEplot(convect2, ic, srange, xrange, trange, basechar=only,
   numchar=50, numsteps=[0,20]);
```



6. Burgers' Equation

One special feature of the PDE (4.1) is that the unknown function u and its partial derivatives appear linearly. An important consequence is the principle of superposition: if u_1 and u_2 are solutions of Eq. (4.1), so too is any linear combination $au_1 + bu_2$ with a and b constant. The Method of Characteristics, however, is not limited to solving such linear PDEs. As a example of an equation that is not linear, we will first study the PDE introduced by J. M. Burgers as a prototype of the PDEs of fluid dynamics, for which the initial-value problem is

$$\frac{\partial u}{\partial t}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = 0 \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{6.1}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}. \tag{6.2}$$

Contrast Eqs. (4.1) and (6.1): the known convection velocity c has been replaced by the unknown function u is itself. According to the Method of Characteristics, we want to solve, for each fixed $\xi \in \mathbb{R}$, the ordinary differential equation initial-value problem

$$\frac{dx}{dt}(t) = u(x,t) \quad \text{for } t > 0, \tag{6.3}$$

$$x(0) = \xi. \tag{6.4}$$

With $t \mapsto X(\xi,t)$ denoting the solution, the parameterized curve $t \mapsto (X(\xi,t),t)$ in the (x,t)-plane is a characteristic curve.

If u satisfies Eq. (6.1), then U, as defined by

$$U(\xi,t) = u(X(\xi,t),t), \tag{6.5}$$

satisfies

$$\begin{aligned}\frac{\partial U}{\partial t}(\xi,t) &= \frac{\partial u}{\partial t}(X(\xi,t),t) + \frac{\partial u}{\partial x}(X(\xi,t),t)\,\frac{\partial X}{\partial t}(\xi,t)\\ &= \frac{\partial u}{\partial t}(X(\xi,t),t) + \frac{\partial u}{\partial x}(X(\xi,t),t)\,u(X(\xi,t),t)\\ &= 0.\end{aligned}$$

Also, $U(\xi,0) = u(X(\xi,0),0) = u(\xi,0) = u_0(\xi)$. Thus the PDE (6.1) requires u to be constant along characteristic curves, and the initial condition (4.2) determines the constant, just as before.

The trouble is that solving Eq. (6.3) seems to require prior knowledge the unknown solution u. We can address this difficulty by solving the differential equation for U simultaneously with the equation for X. This pair of differential equations is

$$\frac{\partial X}{\partial t}(\xi,t) = U(\xi,t), \tag{6.6}$$

$$\frac{\partial U}{\partial t}(\xi,t) = 0 \tag{6.7}$$

for $t > 0$, and the corresponding pair of initial conditions is

$$X(\xi,0) = \xi, \tag{6.8}$$

$$U(\xi,0) = u_0(\xi). \tag{6.9}$$

Remember that we can regard ξ as fixed, so that Eqs. (6.6) and (6.7) can be treated as ordinary differential equations.

This initial-value problem is particularly easy to solve. Indeed, Eq. (6.7) implies that $U(\xi,t)$ is independent of time; in fact, by Eq. (6.9),

$$U(\xi,t) = u_0(\xi). \tag{6.10}$$

The right-hand side of Eq. (6.6) is therefore a constant (as far as t is concerned), so that the solution of Eq. (6.6) is found to be

$$X(\xi,t) = u_0(\xi)\,t + \xi \tag{6.11}$$

after taking Eq. (6.8) into account.

To complete the construction of a solution, we solve the equation

$$x = u_0(\xi)\,t + \xi \tag{6.12}$$

for ξ in terms of x and t and substitute for ξ in

$$u(x,t) = u_0(\xi). \tag{6.13}$$

Example 6.1. As initial data for Burgers' equation let us take

$$u_0(x) = \begin{cases} -1 & \text{if } x < -1,\\ x & \text{if } -1 \le x \le 1,\\ 1 & \text{if } 1 < x.\end{cases} \tag{6.14}$$

Then Eq. (6.12) becomes

$$x = \begin{cases} -t + \xi & \text{if } \xi < -1, \\ \xi t + \xi & \text{if } -1 \le \xi \le 1, \\ t + \xi & \text{if } 1 < \xi. \end{cases} \tag{6.15}$$

Solving gives

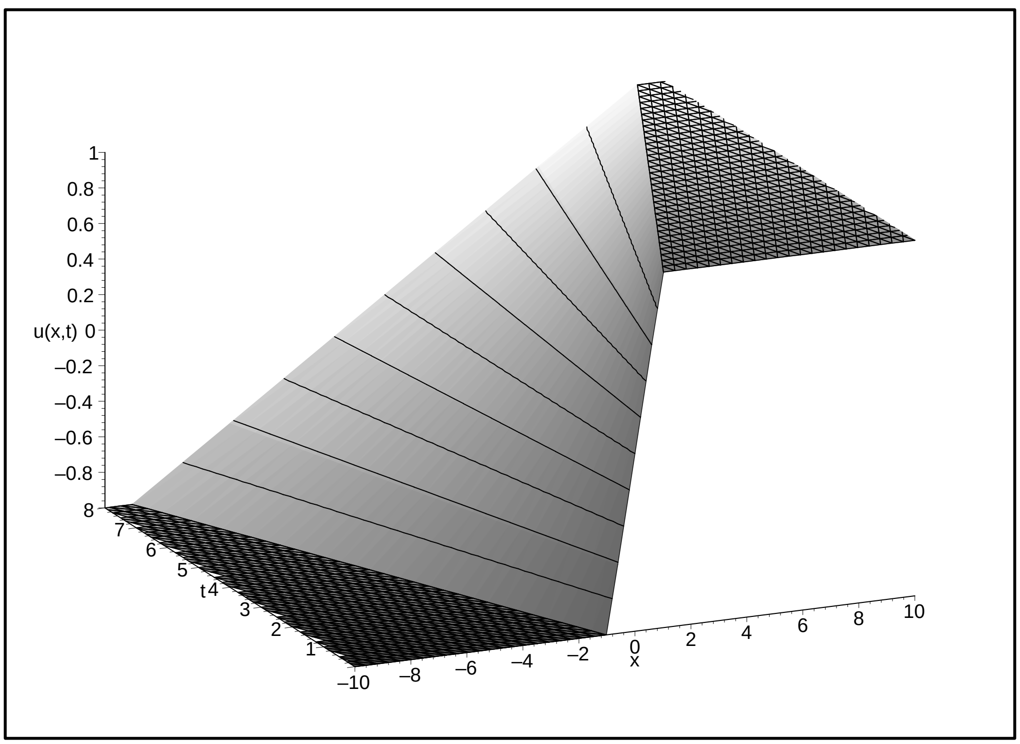
$$\xi = \begin{cases} x + t & \text{if } x < -1 - t, \\ x/(1+t) & \text{if } -1 - t \le x \le 1 + t, \\ x - t & \text{if } 1 + t < x, \end{cases} \tag{6.16}$$

and substituting into Eq. (6.13) yields

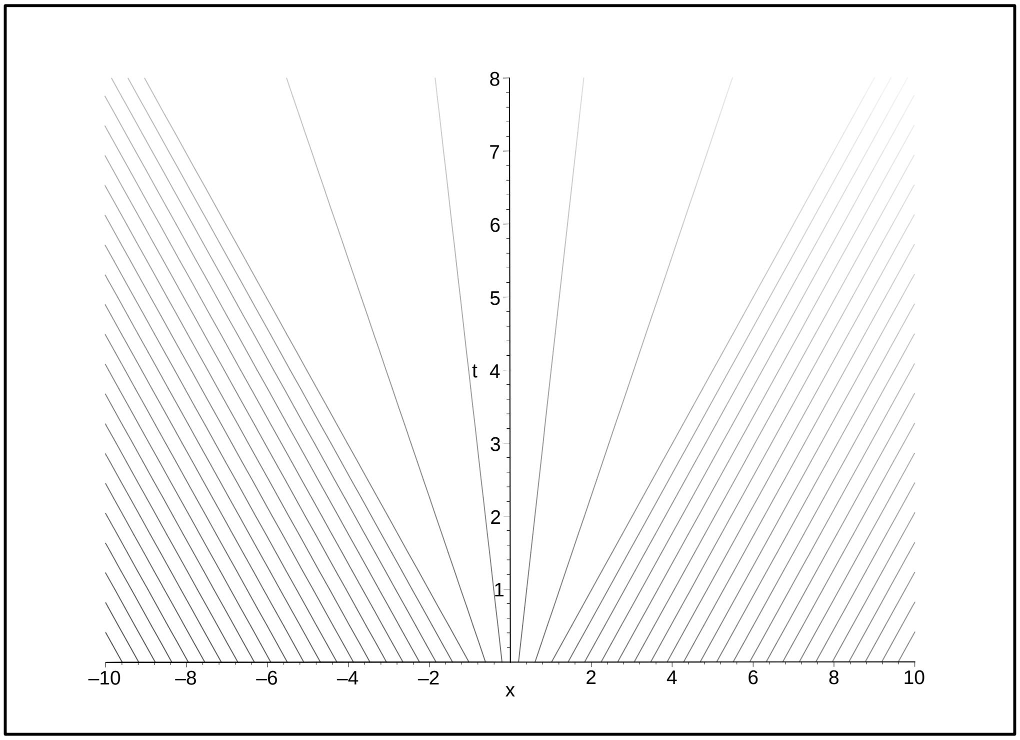
$$u(x,t) = \begin{cases} -1 & \text{if } x < -1 - t, \\ x/(1+t) & \text{if } -1 - t \le x \le 1 + t, \\ 1 & \text{if } 1 + t < x. \end{cases} \tag{6.17}$$

We can also use *Maple* to draw this solution.

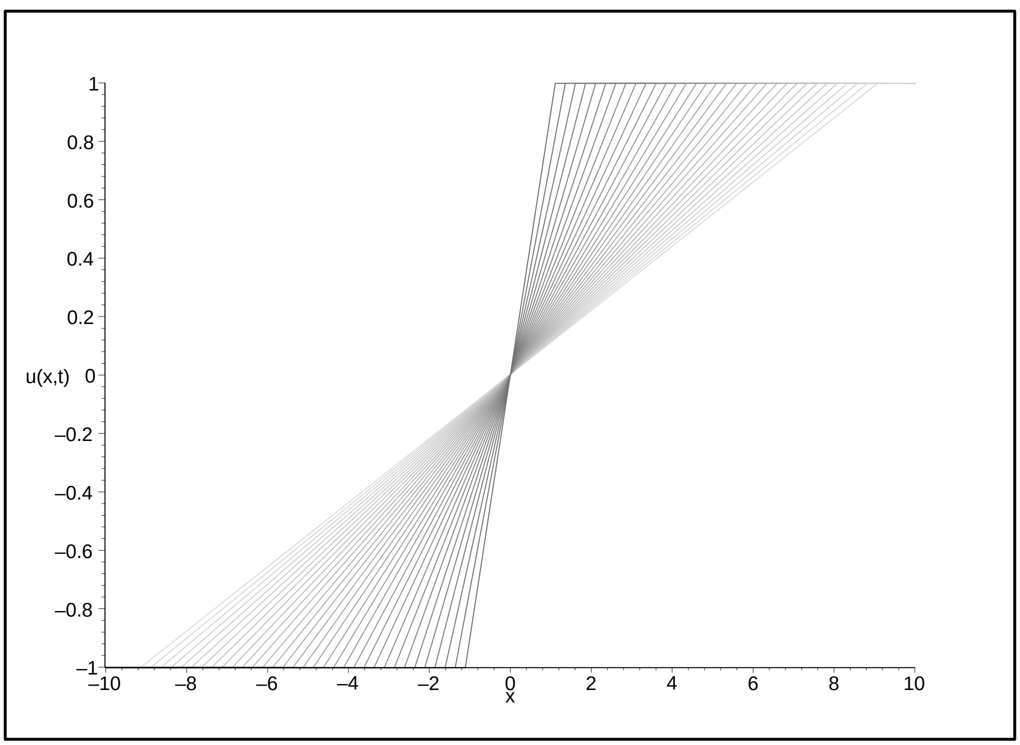
```
> burgers := diff(u(x,t),t) + u(x,t)*diff(u(x,t),x) = 0:
  ic := [s, 0, piecewise(s < -1, -1, 1 < s, 1, s)]:
  srange := s=-10..10:
  xrange := x=-10..10:
  trange := t=0..8:
> PDEplot(burgers, ic, srange, xrange, trange,
  style=patchcontour, orientation=[-114,64], numchar=50, numsteps=[0,40]);
```




```
> PDEplot(burgers, ic, srange, xrange, trange,
  basechar=only, orientation=[-90,0], axes=normal, numchar=50, numsteps=[0,40]);
```




```
> PDEplot(burgers, ic, srange, xrange, trange,
  style=wireframe, orientation=[-90,90], numchar=10, numsteps=[0,40]);
```



Thus the solution spreads out in time. In essence, the higher values of u on the right move toward the right with greater velocity (namely, u). In analogy with waves in gas dynamics that spread out, or rarefy, this kind of solution is called a *rarefaction wave*.

Example 6.2. Let us consider another choice of initial data for Burgers' equation:

$$u_0(x) = \begin{cases} 1 & \text{if } x < -1, \\ -x & \text{if } -1 \le x \le 1, \\ -1 & \text{if } 1 < x. \end{cases} \tag{6.18}$$

Then Eq. (6.12) becomes

$$x = \begin{cases} t + \xi & \text{if } \xi < -1, \\ -\xi t + \xi & \text{if } -1 \le \xi \le 1, \\ -t + \xi & \text{if } 1 < \xi. \end{cases} \tag{6.19}$$

Equation (6.19) reveals a problem: when $t > 1$, x fails to be monotonic in ξ, so that the relationship between x and ξ cannot be inverted. By contrast, in the previous example (see Eq. (6.15)), x is a strictly increasing function of ξ for all $t \ge 0$.

If we restrict t to be less than 1, solving for ξ gives

$$\xi = \begin{cases} x - t & \text{if } x < -1 + t, \\ x/(1-t) & \text{if } -1 \le x/(1-t) \le 1, \\ x + t & \text{if } 1 - t < x, \end{cases} \tag{6.20}$$

and substituting into Eq. (6.13) yields

$$u(x,t) = \begin{cases} 1 & \text{if } x < -1 + t, \\ -x/(1-t) & \text{if } -1 + t \le x \le 1 - t, \\ -1 & \text{if } 1 - t < x. \end{cases} \tag{6.21}$$

But if $t \ge 1$, do not have a viable expression for $u(x,t)$. Thus the Method of Characteristics breaks down at $t = 1$.

What does *Maple* do with this problem? It represents the solution not as a function u of x and t but rather as a two-dimensional surface in (x,t,u)-space parameterized by ξ and t:

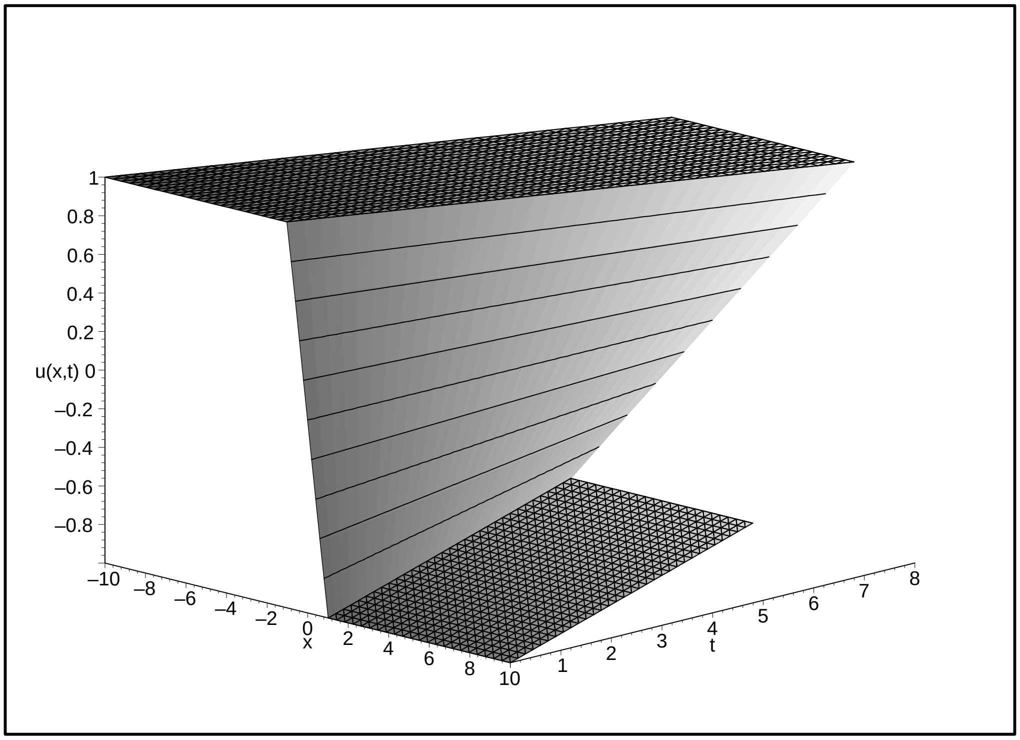
$$x = u_0(\xi)\, t + \xi, \tag{6.22}$$

$$t = t, \tag{6.23}$$

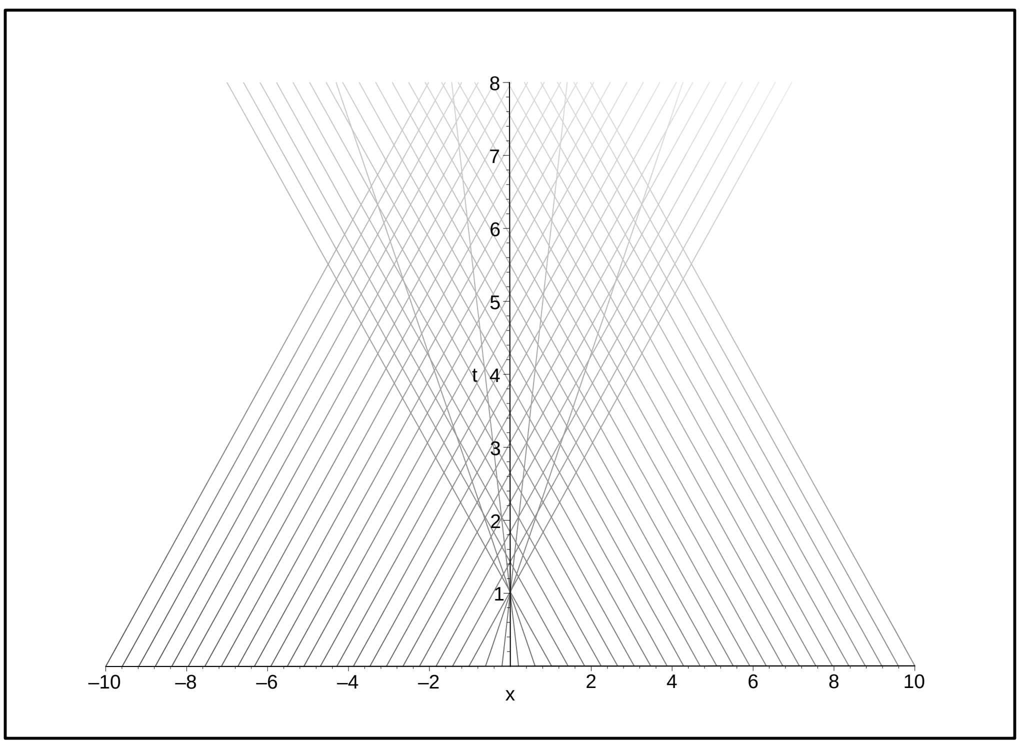
$$u = u_0(\xi). \tag{6.24}$$

This point of view avoids the need to solve for ξ as a function of x and t, which is impossible in this example because the characteristic curves cross.

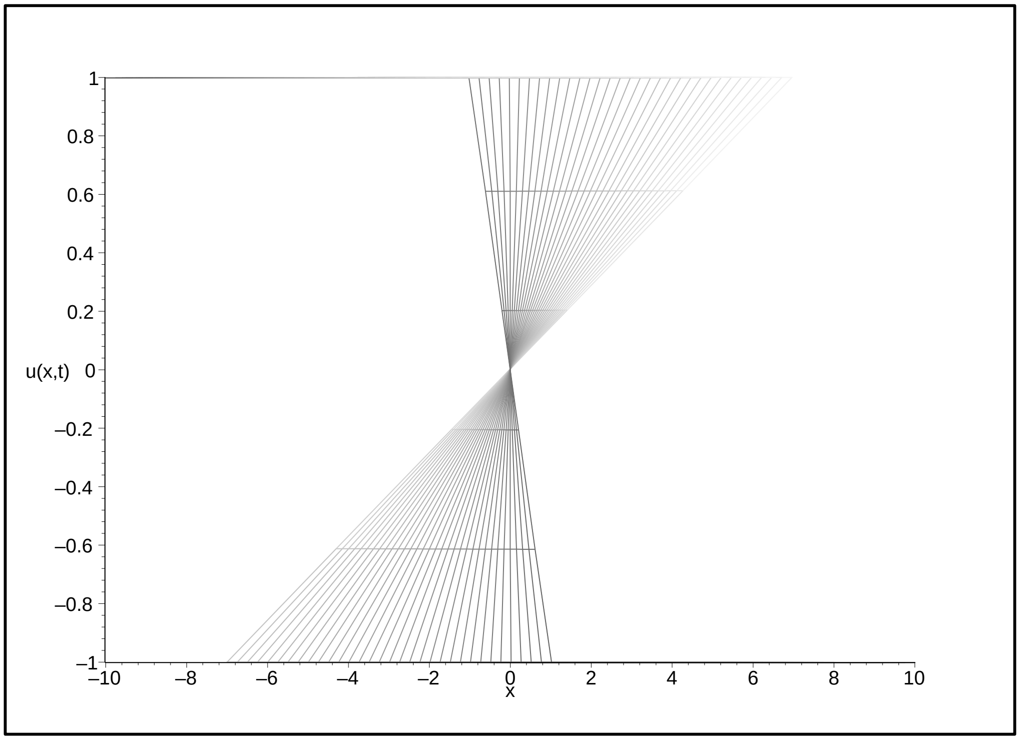
```
> burgers := diff(u(x,t),t) + u(x,t)*diff(u(x,t),x) = 0:
  ic := [s, 0, piecewise(s < -1, 1, 1 < s, -1, -s)]:
  srange := s=-10..10:
  xrange := x=-10..10:
  trange := t=0..8:

> PDEplot(burgers, ic, srange, xrange, trange,
  style=patchcontour, orientation=[-45,70], numchar=50, numsteps=[0,40]);
```




```
> PDEplot(burgers, ic, srange, xrange, trange,
  basechar=only, orientation=[-90,0], axes=normal, numchar=50, numsteps=[0,40]);
```



```
> PDEplot(burgers, ic, srange, xrange, trange,
  style=wireframe, orientation=[-90,90], numchar=50, numsteps=[0,40]);
```



Plots of u *vs.* x for various fixed values of t, however, show a graph that steepens and then becomes multi-valued: The steepening results because the higher values of u on the left move toward the right with greater velocity (namely, u). We will see shortly that this wave-steepening phenomenon can be related to the physical phenomenon of shock wave formation.

7. Shock Waves

Equations like Burgers' equation arise in the modeling of many physical phenomena (for example, traffic flow, as described below). In the context of these models, $u(x,t)$ represents a physical quantity (such as the concentration of cars) at a particular position x and time t. Such physical quantities are not multi-valued! Therefore viewing the solution of Burgers' equation as a surface as a surface in (x,t,u)-space, with multiple values for u corresponding to certain pairs (x,t), is not realistic.

What really happens when characteristic curves cross and the Method of Characteristics says the solution is multi-valued?

To find out, let us rewrite Burgers' equation in the form

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial f(u)}{\partial x}(x,t) = 0, \tag{7.1}$$

where

$$f(u) = \tfrac{1}{2}u^2 \tag{7.2}$$

is called the *flux function*. Consider an interval $[x_1, x_2]$ along the x-axis and an interval $[t_1, t_2]$ along the t-axis. Integrating the previous equation from x_1 to x_2 with respect to x and from t_1 to t_2 with respect to t gives

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \frac{\partial u}{\partial t}(x,t)\, dt\, dx + \int_{t_1}^{t_2} \int_{x_1}^{x_2} \frac{\partial f(u)}{\partial x}(x,t)\, dx\, dt = 0. \tag{7.3}$$

Applying the fundamental theorem of calculus to the integral with respect to t in the first integral and to the integral with respect to x in the second integral yields

$$\int_{x_1}^{x_2} u(x,t_2)\, dx - \int_{x_1}^{x_2} u(x,t_1)\, dx + \int_{t_1}^{t_2} f(u(x_2,t))\, dt - \int_{t_1}^{t_2} f(u(x_1,t))\, dt = 0. \tag{7.4}$$

This key identity is a *conservation principle*, as we now explain. On the one hand, the integral

$$\int_{x_1}^{x_2} u(x,t_*)\, dx \tag{7.5}$$

is the *total amount of "u" in the space interval* $[x_1, x_2]$ *at time* t_*. For example, if u is mass density (with units of mass per unit length), then the integral is the total mass in the space interval $[x_1, x_2]$ at time t_*. On the other hand, the integral

$$\int_{t_1}^{t_2} f(u(x_*,t))\, dt \tag{7.6}$$

is the *total flux of "u" during the time interval* $[t_1, t_2]$ *from left to right past the position* x_*. Again, if u is mass density, then $f(u)$ is the mass flux (that is, the rate of flow of mass, with units of mass per unit time) and the integral is is the total amount of mass that flows past x_* during the time interval $[t_1, t_2]$. Thus Eq. (7.4), which can be written

$$\int_{x_1}^{x_2} u(x,t_2)\, dx = \int_{x_1}^{x_2} u(x,t_1)\, dx - \int_{t_1}^{t_2} f(u(x_2,t))\, dt + \int_{t_1}^{t_2} f(u(x_1,t))\, dt, \tag{7.7}$$

says:

The total amount of "u" in the interval $[x_1, x_2]$ *at time* t_2 *equals the total amount of "u" in the interval* $[x_1, x_2]$ *at time* t_1 *minus the total amount of "u" that flows out of* $[x_1, x_2]$ *at* x_2 *during the time interval* $[t_1, t_2]$ *plus the total amount of "u" that flows into* $[x_1, x_2]$ *at* x_1 *during the time interval* $[t_1, t_2]$.

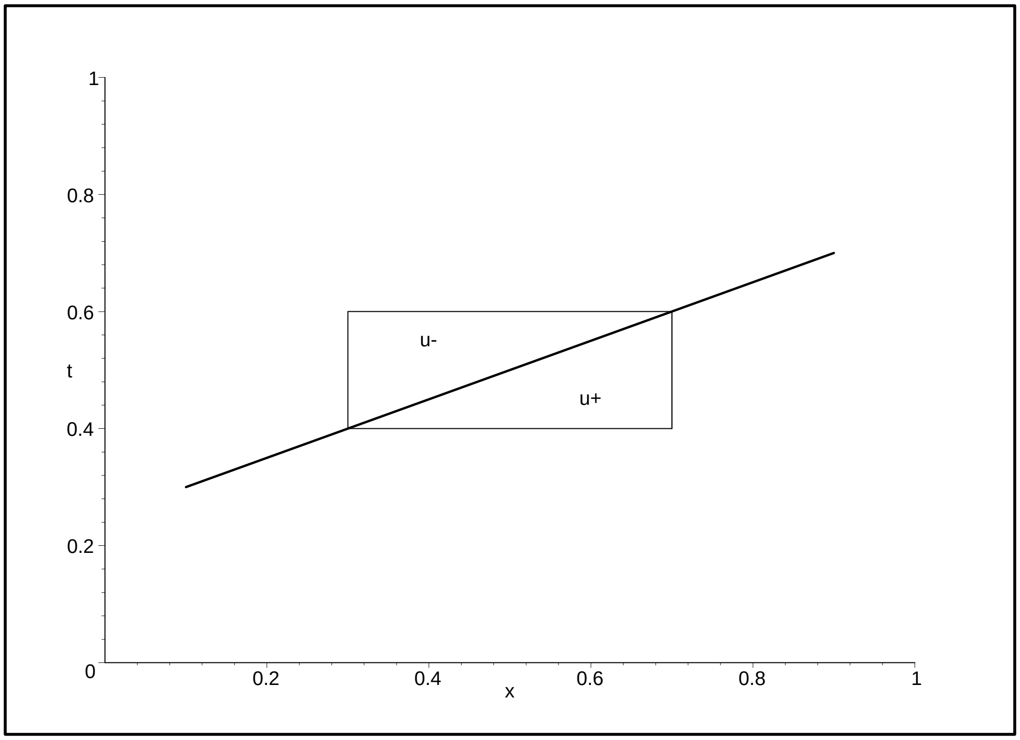
This balance principle, which is eminently reasonable, is the physical content of the PDE (7.1). In fact, the way the PDE (7.1) is usually derived is to assume the physical balance principle, in the form of Eq. (7.4), conclude that Eq. (7.3) holds for arbitrary intervals $[x_1, x_2]$ and $[t_1, t_2]$, and deduce Eq. (7.1) (by taking x_2 close to x_1 and t_2 close to t_1). Notice, however, that the balance principle makes sense in situations where the PDE (7.1) does not, such as when the function u is not differentiable. When the PDE (7.1) is not applicable, the balance principle is more fundamental as a physical law.

Adopting the balance principle as fundamental permits us to consider "solutions" u that have jump discontinuities. Such a function u is not differentiable, and therefore cannot satisfy Eq. (7.1) in the ordinary sense, but it does satisfy the balance principle. These discontinuities are often called *shock waves*. For example, the sonic boom generated by a supersonic airplane is a shock wave in the air. Referring to the question posed at the beginning of this section, what happens when characteristic curves cross is that shock waves form.

A particularly simple kind of shock wave solution takes the form

$$u(x,t) = \begin{cases} u_- & \text{if } x < st, \\ u_+ & \text{if } st < x, \end{cases} \tag{7.8}$$

where u_-, u_+, and s are constants. The quantity s represents the constant velocity at which the shock wave moves. Let us see what restrictions the balance principle places u_-, u_+, and s. Refer to the following figure, the thick line represents the shock trajectory $x = st$, with $s = .5$, and the box is defined by the space and time intervals $[x_1, x_2] = [.3, .7]$ and $[t_1, t_2] = [.4, .6]$.



For this situation,

$$\int_{x_1}^{x_2} u(x, t_1)\, dx = u_+ \Delta x, \tag{7.9}$$

$$\int_{x_1}^{x_2} u(x, t_2)\, dx = u_- \Delta x, \tag{7.10}$$

$$\int_{t_1}^{t_2} f(u(x_1, t))\, dt = f(u_-)\Delta t, \tag{7.11}$$

$$\int_{t_1}^{t_2} f(u(x_2, t))\, dt = f(u_+)\Delta t, \tag{7.12}$$

where $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$. Since $\Delta x / \Delta t = s$, the balance law (7.4), divided by Δt, says that

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0. \tag{7.13}$$

This is the *Rankine-Hugoniot jump condition*, which is the balance principle for a shock wave. More generally, the Rankine-Hugoniot condition holds at every point along a curved shock trajectory (as follows by the preceding argument by taking x_2 close to x_1 and t_2 close to t_1).

8. Numerical Methods

As we have seen, the Method of Characteristics (and the `PDEplot` procedure in *Maple* based on it) constructs solutions of PDE (7.1) that become multi-valued when characteristic curves cross. However, other ways of solving Eq. (7.1) allow for the formation of shock waves. Such methods are based on *numerical approximations* of the balance principle. A full description of numerical methods for Eq. (7.1) is beyond the scope of these notes, but the basic idea can be described easily.

We divide the (x,t)-plane into small rectangles of width Δx and height Δt. Consider one of these rectangles, $[x_1, x_2] \times [t_1, t_2]$. Let us denote the average value of u over a horizontal edge of the rectangle by $\overline{u}$:

$$\overline{u}_{\text{top}} = \frac{1}{\Delta x}\int_{x_1}^{x_2} u(x,t_2)\,dx, \tag{8.1}$$

$$\overline{u}_{\text{bottom}} = \frac{1}{\Delta x}\int_{x_1}^{x_2} u(x,t_1)\,dx. \tag{8.2}$$

If Δx is small, then an average over a horizontal edge is a good approximation to the value of u on that edge. Let us also denote by $\overline{f(u)}$ the average of $f(u)$ over a vertical edge:

$$\overline{f(u)}_{\text{left}} = \frac{1}{\Delta t}\int_{t_1}^{t_2} f(u(x_1,t))\,dt, \tag{8.3}$$

$$\overline{f(u)}_{\text{right}} = \frac{1}{\Delta t}\int_{t_1}^{t_2} f(u(x_2,t))\,dt. \tag{8.4}$$

Then the balance principle (7.4) requires that

$$\overline{u}_{\text{top}} = \overline{u}_{\text{bottom}} + \frac{\Delta t}{\Delta x}\left[\overline{f(u)}_{\text{right}} - \overline{f(u)}_{\text{left}}\right]. \tag{8.5}$$

This identity tells us how to compute $\overline{u}_{\text{top}}$ (that is, the average of u at a later time) provided that we know $\overline{u}_{\text{bottom}}$ (that is, the average of u at an earlier time) together with $\overline{f(u)}_{\text{left}}$ and $\overline{f(u)}_{\text{right}}$. Since we know u at the initial time, we can apply the identity successively to find $\overline{u}$ along each horizontal edge at a later time. The key issue is: How do we determine the average $\overline{f(u)}$ over vertical edges, knowing only $\overline{u}$ along the horizontal edges at earlier times? At this stage an approximation must be made, since $\overline{f(u)}$ cannot be determined exactly from this limited information. There are many choices of a "scheme" for making the approximation, and much research effort has been directed at devising good schemes.

The following is a *Maple* worksheet that uses a scheme called *Godunov's scheme* to solve Burgers' equation.

```
> with(plots):
```

The procedures for solving conservation laws are contained in the following files.

```
> read grid:
  read field:
```

```
read scheme:
read evolve:

grid procedures: create_grid, get_n_cells_of_grid, get_spacing_of_grid

field procedures: create_field, initialize_field, Gaussian_func, advance_field,
get_profile_of_field, get_profile_of_field_at_time

scheme procedures:  CIR_stencil, CIR_l_bc, CIR_r_bc, CIR_max_time_step,
LF_stencil, LF_l_bc, LF_r_bc, LF_max_time_step,
LW_stencil, LW_l_bc, LW_r_bc, LW_max_time_step,
Godunov_stencil, Godunov_l_bc, Godunov_r_bc, Godunov_max_time_step,
Osher_stencil, Osher_l_bc, Osher_r_bc, Osher_max_time_step

scheme global variables:  CFL_factor, artificial_viscosity,
Riemann_solver, Osher_flux,
CIR_scheme, LF_scheme, LW_scheme, Godunov_scheme, Osher_scheme

evolve procedures:  evolve, get_animation, get_3d_plot
```

The function `flux` defines the PDE to be solved. The function `velocity` is its derivative.

```
> flux := u -> .5*u^2:
  velocity := u -> u:
```

The Courant-Friedrich-Levy (CFL) number is a positive nondimensional ratio that must be less than 1 for the scheme to be stable. The Godunov scheme is based upon solving Riemann initial-value problems. Here we set the Riemann solver to be the one appropriate for Burgers' equation.

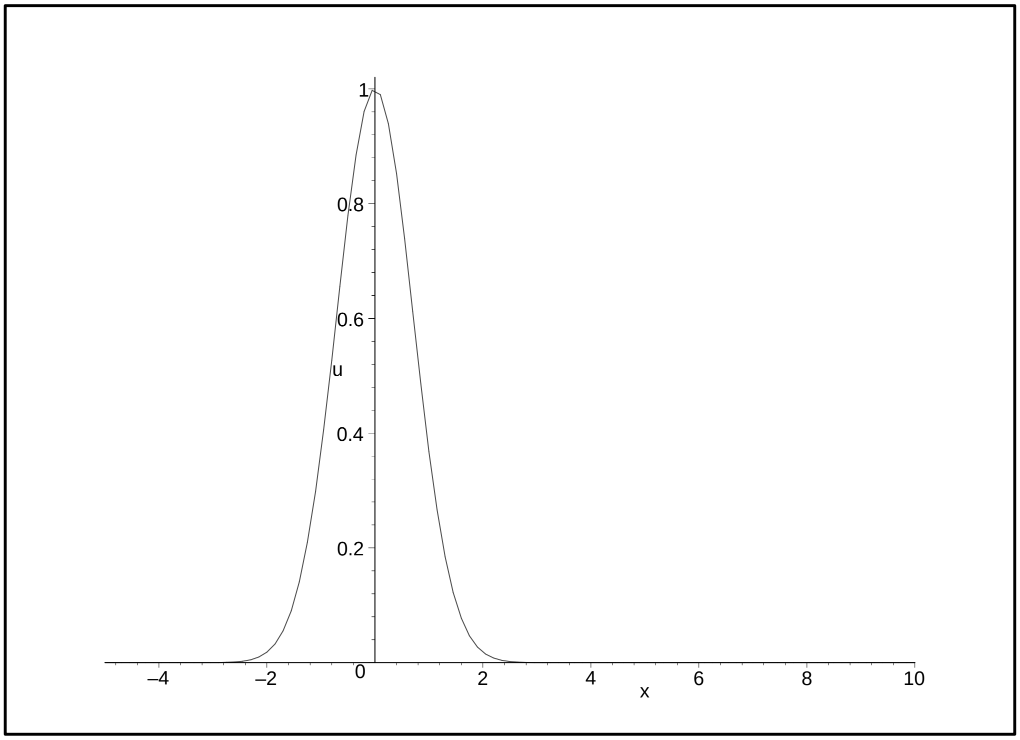
```
> CFL_factor := .9:
  Riemann_solver := Burgers_Riemann_solver:
```

The following lines set up the computational grid.

```
> x_min := -5.:
  x_max := 10.:
  n_cells := 100:
  g := create_grid(x_min, x_max, n_cells):
  t_min := 0.:
  t_max := 8.:
  plot_freq := 1:
```

Next we create and initialize the computational field u, which approximates the solution. The initial data function is a Gaussian profile.

```
> u := create_field(g):
  center := 0.:
  width := 1.:
  amplitude := 1.:
  initialize_field(u, g, Gaussian_func(center, width, amplitude)):
  x_range := 'x'=x_min..x_max:
  u_range := 'u'=0..1.02*amplitude:
  plot(get_profile_of_field(u, g), x_range, u_range);
```

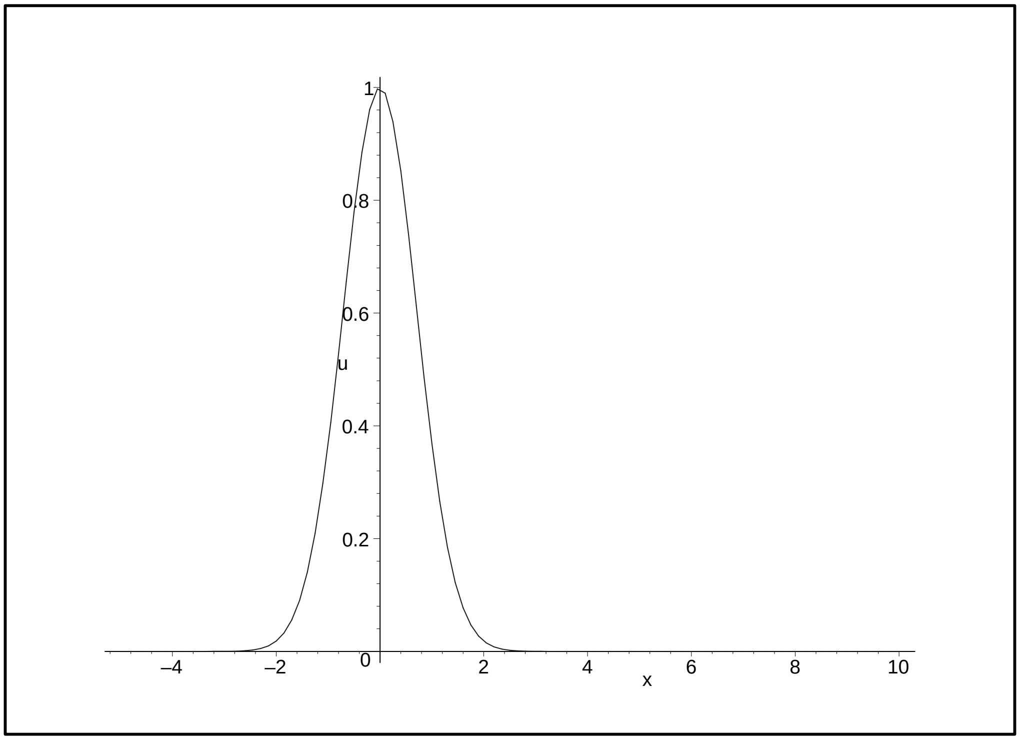


The next line carries out the time evolution of the field u.

```
> data := evolve(u, g, Godunov_scheme, t_min, t_max, plot_freq):
```

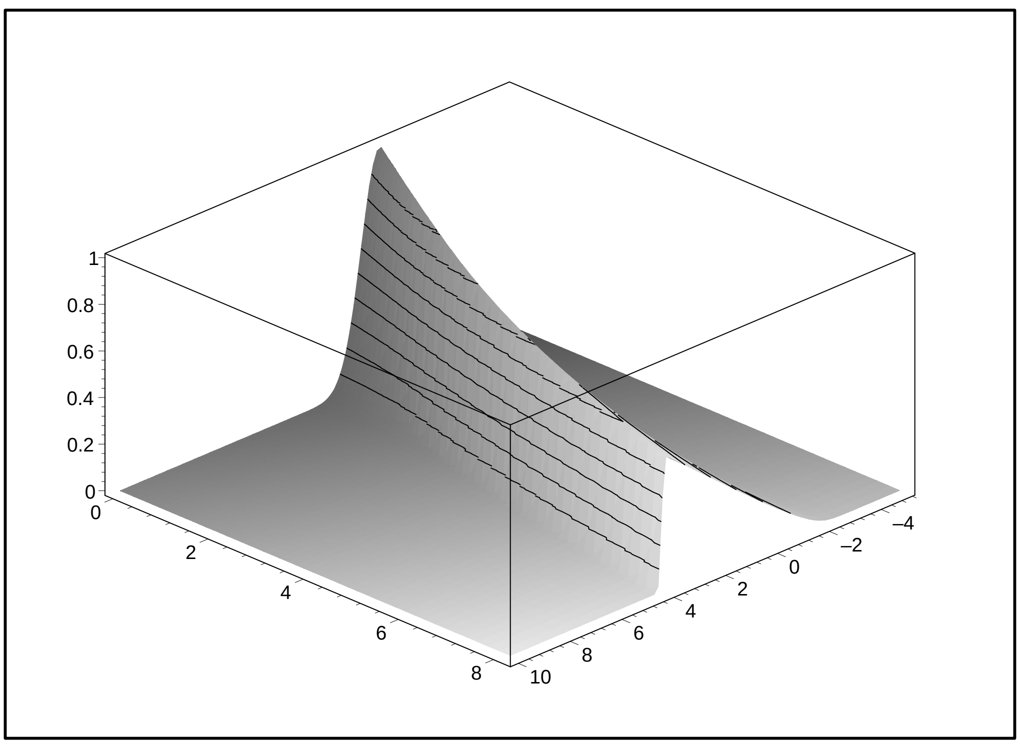
From the output data we can construct an animation of the time evolution.

```
> get_animation(data);
```



From the output data we can also construct a three-dimensional plot of the time evolution.

```
> get_3d_plot(data);
```



Running the animation and examining the three-dimensional plot of the solution reveals the following facts:

- During the early stages of the evolution of the solution, the Gaussian profile leans progressively further to the right. This happens because the characteristic velocity, namely u, is larger at the peak than at the base. The side of the profile to the right of the peak steepens, and while the left side broadens.
- At a certain time, a portion of the profile, on the right side of the peak, becomes vertical. This vertical portion is a jump discontinuity in the solution, that is, a shock wave. The appearance of a discontinuity is called shock wave formation.
- In the meantime, the side of profile to the left spreads out. This part of the profile is a rarefaction wave.
- For a period of time after shock wave formation, the height of the jump (that is, the "strength" of the shock wave) grows. Eventually, however, the shock wave reaches maximum strength.
- At this stage, the solution comprises a ramp on the left (the rarefaction wave) adjoining a discontinuity on the right (the shock wave). This composite configuration of waves is a called an N-wave. As time progresses, the rarefaction and shock waves interact in a way that causes the strength of the shock wave to decay slowly. In effect, the rarefaction and shock waves cancel each other.

It should be emphasized that the numerical solution, which is only an approximation, does not perfectly reproduce the exact solution. In particular, the shock wave is not an exact discontinuity, and the tail of the rarefaction wave is more rounded than in the exact solution.

9. Traffic Dynamics

The ideas discussed in the preceding sections have application to traffic flow along a long road. A simple model was developed by Lighthill and Whitham [1] and by Richards [2]; it is discussed more extensively in Ref. [3].

Let the position along the road (measured in miles) be denoted z, and let ρ denote the density of cars, the number of cars per mile of road. There is a maximum density, $\rho_{\max}$, that the road can accommodate, so that $0 \leq \rho \leq \rho_{\max}$. A typical value for a single-lane road is $\rho_{\max} = 225$ cars per mile.

The flux of cars (that is, the flow rate of cars, measured in cars per hour) can be written ρv, where v is the flow velocity of the traffic (measured in miles per hour). The conservation principle that *cars are neither created or destroyed* requires that

$$\int_{z_1}^{z_2} \rho(z,t_2)\,dz - \int_{z_1}^{z_2} \rho(z,t_1)\,dz + \int_{t_1}^{t_2} (\rho v)(z_2,t)\,dt - \int_{t_1}^{t_2} (\rho v)(z_1,t)\,dt = 0 \tag{9.1}$$

for each interval $[z_1, z_2]$ along the road and each time interval $[t_1, t_2]$. We are assuming, for simplicity, that there are no entrances to or exits from the road. (If there were, the right-hand side of the conservation principle would have to be modified, with zero replaced by a *source term.*) As outlined in Section 7, the conservation principle implies that the PDE

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial z} = 0 \tag{9.2}$$

is satisfied, provided that the solution is sufficiently smooth (in particular, there are no shock waves).

At this point of the modeling process, we have two unknown functions of z and t, namely ρ and v; but we have only one equation (the PDE). Further information or assumption is needed. The simplest model assumes that the flux ρv can be written directly as a function of ρ itself. Such a function is called a *constitutive relation* or *equation of state*. (Names from thermodynamics are adopted because an analogous thermodynamic equation is used to complement the conservation laws governing the flow of a gas.)

What should the constitutive relation look like? Let us first focus on the question of how v should vary with ρ. When ρ is small, the road is nearly empty, the few cars that occupy the road can move at a maximum velocity (say, the speed limit, $v_{\max}$). Let us assume that, as the road becomes more crowded, drivers slow down out of prudence, and that when the density nears $\rho_{\max}$, drivers slow to zero velocity. Thus we are led to assume that the velocity decreases monotonically from a maximum $v_{\max}$ at $\rho = 0$ to zero at $\rho = \rho_{\max}$. For concreteness, we will model this relationship, at least qualitatively, with a linear function:

$$v = v_{\max}(1 - \rho/\rho_{\max}). \tag{9.3}$$

Therefore the flux function is

$$\rho v = \rho v_{\max}(1 - \rho/\rho_{\max}). \tag{9.4}$$

This flux function is zero at $\rho = 0$ and at $\rho = \rho_{\max}$; it has a maximum at $\rho = \frac{1}{2}\rho_{\max}$, when the flow velocity is $v = \frac{1}{2}v_{\max}$. (For typical traffic, the maximum occurs at a somewhat smaller density, about 80 cars per mile, at which the flux is about 1500 cars per hour; this corresponds to a flow velocity of about 20 miles per hour.) Now we have a complete set of equations.

To simplify the problem and to isolate the essential parameters, it is useful to nondimensionalize the equation. To this end, let us introduce the variable $u = \rho/\rho_{\max}$, which is the *concentration of cars*. We have that $0 \le u \le 1$. Also, let us introduce $x = z/v_{\max}$. Then simple algebra shows that the PDE becomes

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{9.5}$$

where

$$f(u) = u(1 - u). \tag{9.6}$$

This traffic dynamics equation can be solved using the Method of Characteristics and using numerical schemes, just as can Burgers' equation.

Exercise 9.1. Rewrite the traffic dynamics in the form

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0. \tag{9.7}$$

What is the expression for $a(u)$?

Exercise 9.2. Write equations for characteristic curves for the traffic dynamics.

Exercise 9.3. Consider that a simple shock wave solution for the traffic dynamics equations:

$$u(x,t) = \begin{cases} u_- & \text{if } x < st, \\ u_+ & \text{if } st < x, \end{cases} \tag{9.8}$$

Find an expression for the shock velocity s in terms of u_- and u_+. *Hint*: Use the Rankine-Hugoniot relation to express s as a rational function, and then simplify this expression.

Exercise 9.4. Modify the *Maple* worksheet of Section 8 to solve the traffic dynamics equation. To do this, change the formulae for the variables `flux` and `velocity`, and set the variable `Riemann_solver` to `traffic_Riemann_solver`. Experiment with the initial-value problem with the same Gaussian initial data used for Burgers' equation. What are the differences in behavior between the solution of traffic dynamics and the solution for Burgers' equation?

Exercise 9.5. Use the modified worksheet of the previous exercise to solve the following *red-light/green-light* problem. A line of traffic is stopped at a red light at x = 0, and at time $t = 0$ the red light turns green. The line of traffic initially has maximum concentration and lies between $x = -1$ and $x = 0$. (To change the initial data in the *Maple* worksheet, define a *Maple* function `u0` that represents red-light/green-light initial data, and replace the procedure call to `Gaussian_func` appearing as an argument of `initialize_field` with `u0`. Also, the following are good choices: `x_min := -2`, `x_max := 12`, `n_cells := 150`, and `t_max := 10`.) Explain, in everyday terms, what you observe about the solution of this initial-value problem. Do the phenomena you observe correspond to your experience? (*Hint*: Determine what happens to the first and last cars in the line of traffic.)

References

[1] M. J. Lighthill and G. B. Whitham. On kinematic waves: I. Flood movement in long rivers; II. Theory of traffic flow on long crowded roads. *Proc. Roy. Soc. A*, 229:281–345, 1955.

[2] P. I. Richards. Shock waves on the highway. *Oper. Res.*, 4:42–51, 1956.

[3] G. B. Whitham. *Linear and Nonlinear Waves.* John Wiley & Sons, New York, 1974.

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