The cascades route to chaos

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(Dated: October 18, 2009)

The presence of a period-doubling cascade in dynamical systems that depend on a parameter is one of the basic routes to chaos. It is rarely mentioned that there are virtually always infinitely many cascades whenever there is one. We report that for one- and two-dimensional phase space, in the transition from no chaos to chaos – as a parameter is varied – there must be infinitely many cascades under some mild hypotheses. Our meaning of chaos includes the case of chaotic sets which are not attractors. Numerical studies indicate that this result applies to the forced-damped pendulum and the forced Duffing equations, viewing the solutions once each period of the forcing. We further show that in many cases cascades appear in pairs connected (in joint parameter-state space) by an unstable periodic orbit. Paired cascades can be destroyed or created by perturbations, whereas unpaired cascades are conserved under even significant perturbations.

In Fig. 1 as μ increases towards a value $\mu_F \approx 3.57$, one encounters a family of periodic orbits that undergo an infinite sequence of period doublings with the period of these orbits tending to ∞. The appearance of infinitely-many such (period-doubling) cascades is one of the most prominent features observed in the study of parametrized maps. Cascades were first reported by Myrberg in 1962 [1] (*cf.* Fig. 1), and Robert May popularized their existence to a huge scientific audience [2]. They are found in a large variety of contexts: in Raleigh-Bernard convection [3], damped bouncing balls [4], Van der Pol oscillators [5] (*cf.* Fig. 2), vibratory ball milling [6], dust charge fluctuation in plasma systems [7], external optical injection in lasers [8], delay oscillators [12], vibrating damaged structures [13], glow discharge [14], combustion [15], very slow classical Cepheid stars [16], neuron and pancreatic cells [17], dripping faucets [18], bouncing droplets on soap films [19], Belousov-Zhabotinsky reactions [20], and in the cromorn, a medieval musical instrument [21].

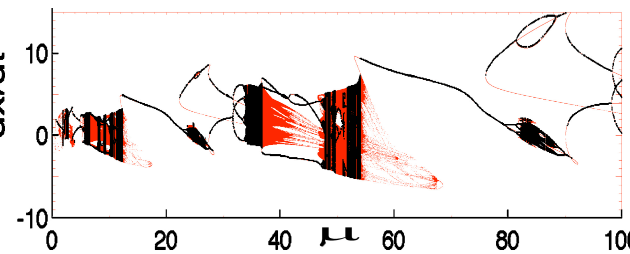

FIG. 2: **Cascades in the double-well Duffing equation.** The attracting sets (in black) and periodic orbits up to period ten (in red) for the time-2π map of the double-well Duffing equation: $x''(t) + 0.3x'(t) - x(t) + (x(t))^2 + (x(t))^3 = \mu \sin t$. Numerical studies show regions of chaos interspersed with regions without chaos, indicating that our Off-On-Off Chaos Theorem applies to this situation.

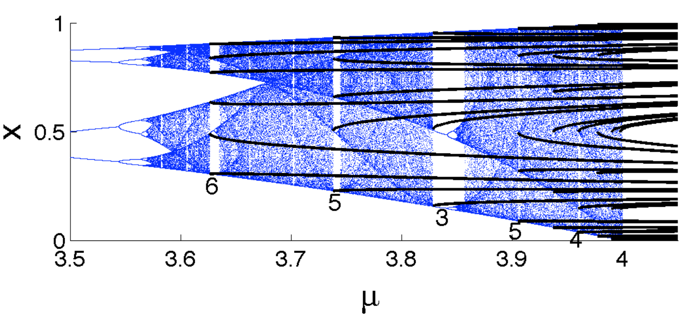

FIG. 1: **Cascades and their connected components.** The attracting set for the logistic map $F(\mu, x) = \mu x(1-x)$ is shown in blue. There are infinitely-many cascades, each with infinitely-many period-doubling bifurcations. Each saddle-node bifurcation creates both a cascade and a path of unstable orbits, the latter shown as black curves for periods up to six. We call the paths of unstable orbits the **stems** of the cascades. Below the largest gaps or *windows* in the blue regions, we give the period of the associated cascade. For this particular map, the black curves of unstable orbits extend to $\mu = +\infty$. Here, at each sufficiently large μ, there is exactly one orbit (k points) in each period-k stem.

We find that for many systems depending on a parameter, every periodic orbit is part of a cascade; for such systems, cascades are as fundamental as periodic orbits themselves. The scaling properties of individual cascades have been studied for cascades in nearly quadratic maps [22], but there are only a few results about the existence of cascades [23]. Furthermore, the mathematical and scientific literature focuses on the study of single cascades. In this Letter, we describe the results of a new general theory of cascades, which explains why cascades exist and why chaotic dynamical systems often have infinitely many cascades.

Two kinds of periodic orbits meet at period-doubling bifurcations. For a map $F(\mu, x)$ that depends on a parameter μ, a point (μ, x_0) is a **period-p point** if $F^p(\mu, x_0) = x_0$, where $p > 0$ is chosen as small as possible. Writing $x_{n+1} = F(\mu, x_n)$, its **(periodic) orbit** is the set of points $\{(\mu, x_0), (\mu, x_1), (\mu, x_2), \cdots, (\mu, x_{p-1})\}$. By the **eigenvalues of** that orbit, we mean the eigenvalues of the

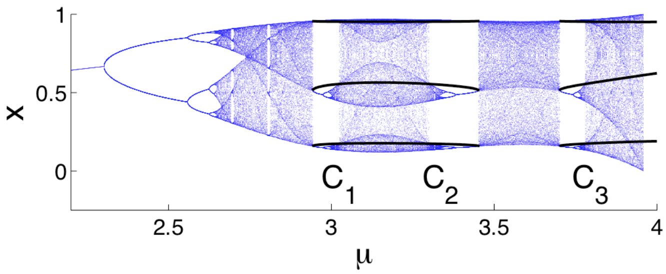

FIG. 3: **Bounded paired cascades.** The attracting set for the function $F(\mu, x) = h(\mu)x(1-x)$, where $h(\mu) = \mu(1.18 + 0.17\cos(2.4\mu))$. Cascades C_1 and C_2 constitute a **pair** of cascades, since they are connected by a family of orbits (black curves). Cascade C_3 is unbounded and unpaired, as indicated by the black curve continuing to the right to $\mu = \infty$.

Jacobian matrix $D_x F^p(\mu, x_0)$. If the map is one-dimensional, then by the derivative of the orbit, we mean the derivative of F^p. A **bifurcation orbit** is any orbit having an eigenvalue with absolute value 1.

We call a periodic orbit a **flip** orbit if the orbit has an odd number of eigenvalues less than -1, and -1 is not an eigenvalue. (In one dimension, this condition is: derivative < -1. In dimension two, flip orbits are those with exactly one eigenvalue < -1.) All other periodic orbits are called **regular**. A family switches between flip and regular orbits at a period-doubling bifurcation orbit since this bifurcation orbit always has an eigenvalue equal to -1.

For ψ in some interval (a, b), let $(\mu(\psi), Y(\psi))$ be a path of *regular* periodic points in (μ, y) space and write $[Y(\psi)]$ to denote the periodic orbit of the periodic point $Y(\psi)$. We say $(\mu(\psi), [Y(\psi)])$ is a **cascade** if the path contains infinitely many period-doubling bifurcations and has orbits with all the periods $\{k, 2k, 4k, 8k, \cdots\}$ for some positive integer k.

Assumptions. *All results in this paper assume that the maps $y \mapsto F(\mu, y)$ are infinitely differentiable jointly in the parameter μ and in phase space y and that all of the periodic orbit bifurcations are* **generic***, meaning that every bifurcation orbit is a standard nondegenerate period-doubling, saddle-node, or Hopf bifurcation.* For example, generic maps have neither symmetries nor symmetry breaking bifurcations. Nor do they have pitchfork or period-tripling bifurcations. We also assume that for any parameter interval $[\mu_1, \mu_2]$, all periodic orbits are contained in a bounded set in phase space.

The cascades route to chaos in dimensions one and two. Usually the term "route to chaos" refers to formation of *chaotic attractors*, in which case there are several possible routes to chaos. In this Letter, we consider routes from a parameter having no chaos to one with *chaotic sets*, where the sets are not necessarily attractors. In fact, we require only one aspect of chaos: we say that there is **chaos** at a particular μ if there are infinitely many regular periodic orbits. For example, there is chaos whenever there is a transverse homoclinic point. That is equivalent to having a horseshoe for some iterate of the map, and it implies there are infinitely many regular saddles in two dimensions, and infinitely many regular unstable orbits in dimension one. This definition of chaos is sufficiently general as to include having one or multiple coexisting chaotic attractors, as well as the case of transient chaos. We say a map has **no chaos** at a particular μ if there are at most finitely many regular periodic orbits.

Cascades Route to Chaos Theorem. *Assume the parametrized map $F(\mu, \cdot)$ has a one- or two-dimensional phase space. If at parameter value μ_1 there is no chaos, and at μ_2, there is chaos and there are at most finitely many attracting orbits (and finitely many repelling orbits in dimension 2), then F has infinitely many cascades between μ_1 and μ_2.*

Thus for one- and two- dimensional smooth families, the only route to chaos is through infinitely many cascades. It is for instance impossible to get chaos at μ_2 from a single cascade.

If we were to omit the assumption that there are infinitely many *regular* orbits, the conclusion would be false. For example, in the quadratic map let μ_F denote the Feigenbaum parameter value, i.e., the first parameter value where the period doublings accumulate. At μ_F there are infinitely many periodic orbits, and all but one are flip orbits. Furthermore μ_F is preceded by a single cascade.

Each cascade will have some minimum period p and will have periodic orbits of periods $p, 2p, 4p, 8p, \cdots$. If the map is one-dimensional or is two-dimensional and dissipative in the sense that there are no repelling periodic orbits, then each of these periods is the period of some attracting orbits in the path, though not all orbits will necessarily be attracting.

Based on numerical studies, a number of maps appear to satisfy the conditions of the Cascades Route to Chaos Theorem. Note that these numerical verifications involve significantly more work than just plotting the attracting sets for each parameter, since we are concerned about both the stable and the unstable behavior to determine whether there is chaos. Examples include the time-2π maps for the double-well Duffing (Fig. 2), the triple-well Duffing, and forced-damped pendulum (Fig. 4), as well as the Ikeda map used to describe the field of a laser cavity, and the pulsed damped rotor map.

We now discuss the creation of a path $(\mu(\psi), [Y(\psi)])$ of *regular orbits*. Consider the procedure of starting at a regular periodic orbit and following the path of periodic orbits containing it in (μ, y)-space, following only regular orbits. Follow the path – such as numerically – through saddle-node bifurcations by reversing the direction in the parameter space. When period-p orbits reach

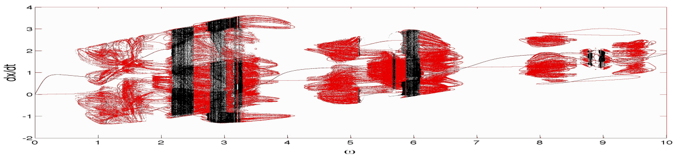

FIG. 4: **The forced-damped pendulum.** For this figure, periodic points with periods < 10 were plotted in red for the time-2π map of the forced-damped pendulum equation: $x''(t) + 0.2x'(t) + \sin(x(t)) = \omega\cos(t)$, indicating the general areas with chaotic dynamics for this map. Then the attracting sets were plotted in black, hiding some periodic points. Much more detailed calculations confirm that interspersed with the chaos, there are some parameter ranges without chaos.

a period-doubling bifurcation $(\mu_0, [Y])$, follow the period-$2p$ orbits. They are regular orbits, whereas the extension past $(\mu_0, [Y])$ of period-p orbits are flip orbits. Likewise, at a period-halving bifurcation, follow along the branch of regular orbits. Following a path away from a cascade can lead to $|\mu| \to \infty$ or it can lead to a second cascade. We call an entire connected path of regular orbits a **component**. Two orbits are in the same component if and only if it is possible to follow a path of regular orbits from one orbit to the other continuously, without jumps, though the period is allowed to double many times.

There are two types of cascades. Only one type of cascades occurs for quadratic maps, but there is a second type of cascade as well. Namely, the component of a cascade either lies in a bounded region of (μ, y)-space, or it does not. Accordingly we call the cascade **bounded** or **unbounded**, respectively, though the adjective really describes the cascade's component. Each cascade in the quadratic map (*cf.* Fig. 1) is unbounded because a branch of periodic orbits extends from each cascade to $\mu = \infty$. The constant period branch that extends to infinity is called the **stem** of the (unbounded) cascade. Thus the theory of quadratic maps is the theory of unbounded cascades. We call the period of the orbits in the stem, the **stem period of the unbounded cascade**. (We only treat the case of an unbounded cascade which has a constant period stem.)

We refer to two cascades as a **pair** if they are in the same component; that is, if the two cascades have a path of periodic orbits running from one the other (*cf.* Fig. 3). We have shown the following rigorous result.

Bounded Paired Cascades Theorem. *Bounded cascades always come in pairs.*

The theorem below shows that paired cascades are common. Once again we consider x in a one- or two-dimensional phase space. We say that map F has **off-on-off chaos** for $\mu_1 < \mu_2 < \mu_3$ if there is no chaos at μ_1 and μ_3, whereas at μ_2, F is as assumed in the Cascades Route to Chaos Theorem.

Off-On-Off Chaos Theorem. *If F is two-dimensional and has off-on-off chaos for $\mu_1 < \mu_2 < \mu_3$, then F has infinitely many bounded paired cascades and at most finitely many unbounded cascades.*

Our numerical studies indicate that the time-2π maps of the forced double-well Duffing (Fig. 2) and forced damped pendulum (Fig. 4) have off-on-off chaos, and that it occurs on multiple non-overlapping parameter regions. Here 2π is the forcing period. In fact we cannot prove there are only finitely many periodic attractors, though we find very few. These systems have no periodic repellers.

Conservation of unbounded cascades. If in addition to the hypotheses of the Cascades Route to Chaos Theorem, we assume further that (i) at μ_1 there are no regular periodic orbits and (ii) at μ_2 there are no attractors (nor repellers in dimension two), then the component of *each* regular orbit at μ_2 contains a cascade that is between μ_1 and μ_2, and distinct regular orbits at μ_2 are in different components. Starting at each regular orbit at μ_2, there is a unique path of regular orbits – initially in the direction of μ_1 – ending in a cascade. Thus the number of cascades hitting the boundary of the parameter interval depends only on the number of regular orbits at μ_2, independent of the behavior of the map between μ_1 and μ_2.

This leads to a heuristic **conservation principle for unbounded cascades**: if $F(\mu, x)$ satisfies property (i) for μ sufficiently negative and property (ii) for μ sufficiently positive, then any perturbation $F(\mu, x) + g(\mu, x)$ such that F dominates g for $|\mu| \to \infty$ has the same set of unbounded cascades.

Three rigorous examples of this idea are encapsulated in the following one-dimensional maps

$$\begin{aligned} F(\mu, x) &= \mu - x^2 + g(\mu, x) \quad \text{(quadratic)}, \\ F(\mu, x) &= \mu x - x^3 + g(\mu, x) \quad \text{(cubic)}, \\ F(\mu, x) &= x^4 - 2\mu x^2 + \mu^2/2 + g(\mu, x) \quad \text{(quartic)}, \end{aligned} \tag{1}$$

where for some real positive β,

$$\begin{aligned} |g(\mu, 0)| &< \beta \quad \text{for all } \mu, \text{ and} \\ |g_x(\mu, x)| &< \beta \quad \text{for all } \mu, x. \end{aligned} \tag{2}$$

For such g, each of the three maps each has no regular periodic orbits for μ sufficiently negative, and for μ sufficiently large has a one-dimensional horseshoe map. The quartic map was chosen so that when g is identically 0 and $\mu > 0$, the graph has two minima, at $(\pm\sqrt{\mu}, -\mu^2/2)$, and the local maximum is at $(0, +\mu^2/2)$. The conditions on g guarantee that it does not significantly affect the periodic orbits when $|\mu|$ is sufficiently large; in particular it does not affect their eigenvalues, so it does not affect the number of period-p regular periodic orbits for large $|\mu|$. Hence one can check that all three maps have no regular periodic orbits for μ very negative and have no attracting periodic orbits for μ very positive, and for sufficiently large $|\mu|$, all periodic orbits are contained in the set $[-2\sqrt{\mu}, 2\sqrt{\mu}]$. Thus the following holds.

k	Quad(k)	Cubic(k)	Quart(k)
1	1	2	2
2	0	1	2
3	1	4	10
4	1	8	28
5	3	24	102
6	4	56	330
7	9	156	1152
8	14	400	4064
9	28	1092	14560
$k \gg 10$	$\sim 2^k/2k$	$\sim 3^k/2k$	$\sim 4^k/2k$

TABLE I: The number of unbounded cascades of stem period k for any large-scale perturbation of the quadratic, cubic, and quartic maps, respectively labeled $Quad(k)$, $Cubic(k)$, and $Quart(k)$.

Conservation of cascades theorem. *For any F of the functions in Eqn. 1, with g chosen as in Eqn. 2, the number of stem-period-k unbounded cascades is independent of the choice of g.*

The number of unbounded cascades in each of these three cases is summarized in our Table. The relative sizes reflect the complexity of the three horseshoe maps; the quadratic map has the complexity of a standard two-branch horseshoe, whereas the cubic has a horseshoe with three branches, and the quartic has four branches.

The conservation principle works for two- and higher-dimensional maps as well. We have shown in [25] that large-scale perturbations of the two-dimensional Hénon map conserve unbounded cascades. In [24], we have shown that there is conservation of unbounded cascades for a coupled system of N quadratic maps.

ACKNOWLEDGEMENTS

Thank you to Safa Motesharrei for his corrections and comments. J.A.Y. was partially supported by NSF Grant DMS-0616585 and NIH Grant R01-HG0294501. E.S. was partially supported by NSF Grants DMS-0639300 and DMS-0907818, and NIH Grant R01-MH79502.

[1] P. Myrberg, J. Math. Pures Appl. (9) **41**, 339 (1962).
[2] R. May, Science **186**, 645 (1974).
[3] H. Yahata, Progr. Theoret. Phys. Suppl. pp. 26–74 (1984). S. M. Cox and I. M. Moroz, Physica D: Nonlinear Phenomena **93**, 1 (1996). A. Libchaber, C. Laroche, and S. Fauve, Le Journal de Physique – Lettres **43**, 211 (1982).
[4] N. B.Tufillaro, T. Abbott, and J. Reilly, *An experimental approach to nonlinear dynamics and chaos* (Addison-Wesley, Redwood City, CA, 1992). N. B. Tufillaro, T. M. Mello, Y. M. Choi, and A. M. Albano, Journal de Physique **47**, 1477 (1986). P. J. Holmes, J. Sound Vibration **84**, 173 (1982). P. Pierański, J. Physique **44**, 573 (1983).
[5] J. Yu, R. Zhange, W. Pan, and L. Schimansky-Geier, Phys. Scr. **78**, 025003, 5 (2008).
[6] H. Huang and J. Pan, Complexity International **2** (1995).
[7] M. P. Bora and D. Sarmah, *Parametric excitation and chaos through dust-charge fluctuation in a dusty plasma* (2007), arXiv:0708.0684v1.
[8] T. B. Simpson, J. M. Liu, A. Gavrielides, V. Kovanis, and P. M. Alsing, Applied Physics Letters **64**, 3539 (1994).
[9] C. Letelliera, M. Bennoud, and G. Martel, Chaos, Solitons and Fractals **33**, 782 (2007).
[10] D. Tang, J. Pujol, and C. Weiss, Phys Rev A **44**, R35 (1991).
[11] G. Martel, M. Bennoud, B. Ortac, T. Chartier, J. Nunzi, and et al., J. Modern Opt. **51**, 85 (2004).
[12] L. Larger, P.-A. Lacourt, S. Poinsot, and M. Hanna, Phys. Rev. Lett. **95**, 043903 (2005).
[13] A. Carpinteri and N. Pugno, J. Appl. Mech. **72**, 511 (2005).
[14] D. D. Sijacic, U. Ebert, and I. Rafatov, Phys. Rev. E **70**, 056220 (2004). T. Braun and et al., Phys. Rev. Lett. **59**, 613 (1987). D. G. Dimitriu and L. M. Ivan, in *PLASMA 2007 International Conference on Research and Applications of Plasmas* (AIP Conference Proceedings, 2008), vol. 993, p. 93.
[15] M. Frankel, V. Roytburd, and G. Sivashinsky, SIAM J. Appl. Math. **54**, 1101 (1994). G. Joulin and G. Sivashinsky, Combust. Sci. Tech. **55**, 83 (1987). S. B. Margolis and B. J. Matkowsky, SIAM J. Appl. Math. **45**, 93 (1985). M. Miklavcic, Combustion Theory and Modelling **11**, 103 (2007).
[16] J. Buchler, in *The Impact of Large-Scale Surveys on Pulsating Star Research* (2000), vol. 203, p. 343. N. Lomb, Astrophysics and Space Science **39**, 447 (1976). T. Aikawa, Astronomy and Astrophysics **371**, 667 (2001).
[17] B. Deng, J. Math. Biol. **38**, 21 (1999). M. Schindler and A. Ajdari, Phys. Rev. Lett. **100**, 044501 (2008).
[18] P. Garstecki, M. J. Fuerstman, and G. M. Whitesides, Phys. Rev. Lett. **94**, 234502 (2005).
[19] T. Gilet and J. W. Bush, Phys. Rev. Lett. **102** (2009).
[20] J. G. Freire, R. J. Field, and J. A. C. Gallas, J. Chem. Phys. **131**, 044105 (2009).
[21] V. Gibiat and M. Castellengo, J. Acoust. Soc. Am. **105**, 940 (1999).
[22] M. J. Feigenbaum, J. Statist. Phys. **21**, 669 (1979). H. Epstein, Communications in Mathematical Physics **106**, 395 (1986). E. Lanford, Bull. Amer. Math. Soc. **6**, 427 (1982).
[23] J. A. Yorke and K. T. Alligood, Bull. Amer. Math. Soc. (N.S.) **9**, 319 (1983). J. Franks, Trans. Amer. Math. Soc. **287**, 275 (1985).
[24] E. Sander and J. A. Yorke, *Period-doubling cascades galore* (2009), arXiv:0903.3613, submitted for publication.
[25] E. Sander and J. A. Yorke, Journal of Fixed Point Theory and Applications, (2009).