9 Finite difference methods for hyperbolic differential equations.

In this section, we will discuss finite difference methods for hyperbolic partial differential equations. First we list a few typical hyperbolic PDEs.

- Advection equations (one-way wave equations).

$$
\begin{aligned}
u_t + a u_x &= f(x,t), \quad 0 < x < 1, \\
u(x,0) &= \eta(x), \qquad \text{IC}, \\
u(0,t) = g_l(t), \quad \text{if} \quad a \ge 0, \quad &\text{or} \quad u(1,t) = g_r(t), \quad \text{if} \quad a \le 0.
\end{aligned}
\tag{9.1}
$$

- Second order linear wave equations:

$$
\begin{aligned}
u_{tt} &= a u_{xx} + f(x,t), \quad 0 < x < 1, \\
u(x,0) &= \eta(x), \qquad \text{IC}, \\
u(0,t) &= g_l(t), \quad u(1,t) = g_r(t)
\end{aligned}
\tag{9.2}
$$

- Linear first order hyperbolic system:

$$
\mathbf{u}_t = A\mathbf{u}_x + \mathbf{f}(x,t) \tag{9.3}
$$

  where $\mathbf{u}$ and $\mathbf{f}$ are two vectors, A is a matrix. The system is *hyperbolic* if A is diagonalizable, $A = TDT^{-1}$, and all eigenvalues of A are real numbers.

- Non-linear hyperbolic system, particularly the conservation laws

$$
\begin{aligned}
u_t + f_x = 0, \quad \textit{e.g.}, \text{ Burger's eqn.}, \quad & u_x + \left(\frac{u^2}{2}\right)_x = 0. \\
\mathbf{u}_t + \mathbf{f}_x + \mathbf{g}_y = 0, \quad & \text{in 2D}.
\end{aligned}
\tag{9.4}
$$

  For non-linear hyperbolic PDEs, shocks (discontinuous solution) can develop even if the initial condition is smooth.

9.1 Boundary conditions

We know the exact solution for the one-way wave equation

$$
\begin{aligned}
u_t + a u_x = 0, &\quad -\infty < x < \infty, \\
u(x,0) = \eta(x), &\quad t > 0
\end{aligned}
$$

is $u(x,t) = \eta(x - at)$.

If the domain is finite, we can also find the exact solution. Consider the model problem

$$u_t + au_x = 0, \quad 0 < x < 1,$$

$$u(x,0) = \eta(x), \quad t > 0, \qquad u(0,t) = g_l(t) \quad \text{if } a > 0.$$

We can use the *characteristic method* to solve the problem. Assume that the solution keeps a constant value along a line (the characteristic). Given a point (x,t), we can trace the solution along the characteristic line $(x+ks, t+s)$. Define

$$z(s) = u(x+ks, t+s) \tag{9.5}$$

along which the solution keeps a constant, that is $z'(s) \equiv 0$. Plug this into the PDE, we can get:

$$z'(s) = u_t + ku_x = 0.$$

which is always true of we take $k = a$. Therefore the solution at $(x+ks, t+s)$ is the same as at (x,t). So we can solve the problem by tracing back until the line hit the boundary. Therefore $u(\bar{x},\bar{t}) = u(x+as, t+s) = u(x-at, 0)$ if $x - at \geq 0$ which means we trace back to the initial condition. If $x - at < 0$, we can only trace back to $x = 0$ or $s = -\bar{x}/a$ and $t = \bar{x}/a$ and the solution is $u(\bar{x},\bar{t}) = u(0, t - \frac{\bar{x}}{a}) = g_l(t - \frac{\bar{x}}{a})$. Therefore the solution for the case $a \geq 0$ can be written as

$$u(x,t) = \begin{cases} \eta(x-at) & \text{if } x \geq at, \\ g_l\left(t - \dfrac{x}{a}\right) & \text{if } x < at. \end{cases} \tag{9.6}$$

Now we can see that we have to prescribe a boundary condition at $x = 0$ but we can not have any boundary condition at $x = 1$. It is important to get correct boundary conditions for hyperbolic problems.

The one-way wave equation is often used as a bench-mark problem for different numerical methods for hyperbolic problems.

Simple numerical methods for hyperbolic problems include

- Lax-Friedrichs method.
- Up-wind scheme.
- Leap-frog method
- Box-scheme
- Lax-Wendroff method
- Crank-Nicholson scheme
- Beam-Warming method.

9.2 Lax-Friedrichs method.

Consider the one-way wave equation $u_t + au_x = 0$. One may want to try the simple finite difference scheme

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{a}{2h}\left(U_{j+1}^k - U_{j-1}^k\right) = 0,$$

$$\text{or} \qquad U_j^{k+1} = U_j^k - \mu\left(U_{j+1}^k - U_{j-1}^k\right),$$

where $\mu = a\Delta t/(2h)$. The scheme has $O(\Delta t + h^2)$ local truncation error. But the method is unconditionally unstable. To see why, we conduct the von Neumann stability analysis to get the growth factor:

$$\begin{aligned} g(\theta) &= 1 - \mu\left(e^{ih\xi} - e^{-ih\xi}\right) \\ &= 1 - \mu\, 2\, i\, \sin(h\xi), \end{aligned}$$

where $\theta = h\xi$. Therefore

$$|g(\theta)|^2 = 1 + 4\mu^2 \sin^2(h\xi) \geq 1,$$

and the scheme is unconditionally unstable.

In the Lax-Friedrichs scheme, we average U_j^k using U_{j-1}^k and U_{j+1}^k to get

$$U_j^{k+1} = \frac{1}{2}\left(U_{j-1}^k + U_{j+1}^k\right) - \mu\left(U_{j+1}^k - U_{j-1}^k\right).$$

The local truncation error has an order of $O(\Delta t + h)$ if $\Delta t \sim h$. The growth factor is

$$\begin{aligned} g(\theta) &= \frac{1}{2}\left(e^{ih\xi} + e^{-ih\xi}\right) + \mu\left(e^{ih\xi} - e^{-ih\xi}\right) \\ &= \cos(h\xi) - 2\mu\sin(h\xi). \end{aligned}$$

Therefore

$$\begin{aligned} |g(\theta)|^2 &= \cos^2(h\xi) + 4\mu^2\sin^2(h\xi) \\ &= 1 - \sin^2(h\xi) + 4\mu^2\sin^2(h\xi) \\ &= 1 - (1 - 4\mu^2)\sin^2(h\xi). \end{aligned}$$

We can conclude that $|g(\theta)| \leq 1$ if $1 - 4\mu^2 \geq 0$ or $1 - (a\Delta t/h)^2 \geq 0$, which implies that $\Delta t \leq h/|a|$. This is called the CFL (Courant-Friedrichs-Lewy) condition.

For the Lax-Friedrichs scheme, we need a numerical boundary condition at $x = 1$ which will be explained later. The Lax-Friedrichs scheme is the basis of several other popular schemes.

9.3 The upwind scheme.

The upwind scheme for $u_t + au_x = 0$ is

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} = \begin{cases} -\frac{a}{h}\left(U_j^k - U_{j-1}^k\right) & \text{if } a \geq 0 \\ -\frac{a}{h}\left(U_{j+1}^k - U_j^k\right) & \text{if } a < 0. \end{cases} \tag{9.7}$$

It is first order accurate in time and in space. To find out the CFL constraint, we conduct the von Neumann stability analysis. The growth factor for the case when $a \geq 0$ is

$$\begin{aligned} g(\theta) &= 1 - \mu\left(1 - e^{-ih\xi}\right) \\ &= 1 - \mu(1 - \cos(h\xi)) - i\mu\sin(h\xi). \end{aligned}$$

Now we investigate the magnitude:

$$\begin{aligned} |g(\theta)|^2 &= (1 - \mu + \mu\cos(h\xi))^2 + \mu^2\sin^2(h\xi) \\ &= (1-\mu)^2 + 2(1-\mu)\mu\cos(h\xi) + \mu^2 \\ &= 1 - 2(1-\mu)\mu(1 - \cos(h\xi)) \\ &= 1 - 4\mu\sin^2(h\xi/2) \end{aligned}$$

Therefore if $1 - \mu \geq 0$, that is $\mu \leq 1$, or $\Delta t \leq h/a$, we have $|g(\theta)| \leq 1$.

Note that for the upwind scheme, no numerical boundary condition is needed; no severe time step restriction since $\Delta t \leq h/a$. If $a = a(x,t)$ is a variable function that does not change the sign, then the CFL condition is

$$0 < \Delta t \leq \frac{h}{\max|a(x,t)|}.$$

However, the upwind scheme is first order in time and in space. Below we discuss some high order schemes.

9.4 The Leap-Frog scheme.

The Leap-Frog scheme for $u_t + au_x = 0$ is:

$$\begin{aligned} &\frac{U_j^{k+1} - U_j^{k-1}}{2\Delta t} + \frac{a}{2h}\left(U_{j+1}^k - U_{j-1}^k\right) = 0, \\ \text{or} \qquad &U_j^{k+1} = U_j^{k-1} - \mu\left(U_{j+1}^k - U_{j-1}^k\right), \end{aligned} \tag{9.8}$$

where $\mu = a\Delta t/(2h)$. The discretization is second order in time and in space. It requires a numerical boundary condition at one end and need U_j^1 to get started. We know that the Leap-Frog scheme is unconditionally unstable for the heat equation. So we should be concerned with the stability through the von Neumann stability analysis. Let

$$U_j^k = e^{ij\xi}, \quad U_j^{k+1} = g(\xi)e^{ij\xi}, \quad U_j^{k-1} = \frac{1}{g(\xi)}\, e^{ij\xi}.$$

Plug in these into the Leap-Frog scheme, we can get

$$g^2 + \mu(e^{ih\xi} - e^{-ih\xi})g - 1 = 0,$$

$$\text{or} \qquad g^2 + 2\mu i \sin(h\xi) - 1 = 0.$$

The solution is

$$g_{\pm} = -i\mu\sin(h\xi) \pm \sqrt{1 - \mu^2\sin^2(h\xi)} \tag{9.9}$$

we distinguish three different cases:

1. If $|\mu| > 1$, then there are such ξ's such that one of $|g_-| > 1$ or $|g_+| > 1$ is true. The scheme is unstable!

2. If $|\mu| < 1$, then $1 - \mu^2\sin^2(h\xi) \geq 0$, we have

   $$|g_{\pm}|^2 = \mu^2\sin^2(h\xi) + 1 - \mu^2\sin^2(h\xi) = 1.$$

   However, since it is two stages method, we have to be careful about the stability. For linear finite difference equation theory, we know that the general solution is

   $$\begin{aligned} U^k &= C_1 g_-^k + C_2 g_+^k \\ |U^k| &\leq \max\{C_1, C_2\}\left(|g_-^k| + |g_+^k|\right) \\ &\leq 2\max\{C_1, C_2\}. \end{aligned}$$

   Therefore, the scheme is called neutral stable according to the definition $\|\mathbf{U}^k\| \leq C_T \sum_{j=0}^{J} \|\mathbf{U}^j\|$.

3. If $|\mu| = 1$, we still have $|g_{\pm}| = 1$. However, we can find ξ such that $\mu\sin(h\xi) = 1$, and $g_+ = g_- = -i$. That is $-i$ is a double root of the characteristic polynomial. Therefore the solution of the finite difference equation has the form

   $$U_j^k = C_1(-i)^k + C_2 k(-i)^k,$$

   where C_1 and C_2, can be complex numbers, are determined from the initial conditions. Therefore there are solutions that $\|\mathbf{U}^k\| \sim k$ which is unstable (slow growing).

Therefore, the Leap-Frog scheme is stable if $\Delta t < \frac{h}{|a|}$. Note that we can use the upwind or other scheme (even unstable one) to initialize the Leap-Frog scheme to get U_j^1.

Note that if $|g(\xi)| < 1$, we call the numerical scheme is dissipative. The Leap-Frog scheme is a non-dissipative scheme.

9.5 Modified equations and numerical diffusion and dispersion.

A modified equation is the PDE that a finite difference equation satisfies exactly at grid points. Consider the upwind method for the advection equation $u_t + au_x = 0$ in the case $a > 0$,

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{a}{h}\left(U_j^k - U_{j-1}^k\right) = 0.$$

The process of deriving the modified PDE is similar to computing the local truncation error, only now we insert $v(x,t)$ into the finite difference equation to derive a PDE that $v(x,t)$ satisfies. Therefore we have

$$\frac{v(x.t + \Delta t) - v(x,t)}{\Delta t} + \frac{a}{h}\left(v(x,t) - v(x-h,t)\right) = 0.$$

Expanding these terms in Taylor series about (x,t) and simplifying gives:

$$v_t + \frac{1}{2}\Delta t\, v_{tt} + \cdots + a\left(v_x - \frac{1}{2}hv_{xx} + \frac{1}{6}h^2 v_{xxx} + \cdots\right) = 0.$$

We can rewrite this as

$$v_t + av_x = \frac{1}{2}(ahv_{xx} - \Delta t v_{tt}) + \frac{1}{6}\left(ah^2 v_{xxx} - (\Delta t)^2 v_{tt}\right) + \cdots$$

This is the PDE that v satisfies. From the equation above, we can get

$$\begin{aligned} v_{tt} &= -av_{xt} + \frac{1}{2}(ahv_{xxt} - \Delta t v_{ttt}) \\ &= -av_{xt} + O(\Delta t, h) \\ &= -a\frac{\partial}{\partial x}\left(-av_x + O(\Delta t, h)\right) \end{aligned}$$

Therefore, the leading term of the modified PDE is

$$v_t + av_x = \frac{1}{2}ah\left(1 - \frac{a\Delta t}{h}\right)v_{xx}. \tag{9.10}$$

This is a advection-diffusion equation. The grid values U_j^n can be viewed as giving a second order accurate approximation to the true solution of this equation (whereas they only give first order accurate approximation to the true solution of the original problem. From modified equation, we can conclude the following:

- The computed solution will smooth out discontinuities because the diffusion term. The second order derivative term is called numerical dissipation, or numerical viscosity.
- If a is a constant, and $\Delta t = h/a$, then $1 - \frac{a\Delta t}{h} = 0$, we have second order accuracy as we see from numerical example.

- We can add the correction term for the second order accuracy to get higher order accurate method. The stability needs to be checked. For example, we get modify the upwind scheme to get a second order scheme:

$$\frac{U_j^{k+1}-U_j^k}{\Delta t}+a\frac{U_j^k-U_{j-1}^k}{h}=\frac{1}{2}ah\left(1-\frac{a\Delta t}{h}\right)\frac{U_{j-1}^k-2U_j^k+U_{j+1}^k}{h^2}$$

which is second order accurate if $\Delta t \sim h$.

- From the modified equation, we can see why some scheme is unstable. For example, the leading term of the modified PDE for the unstable scheme

$$\frac{U_j^{k+1}-U_j^k}{\Delta t}+a\frac{U_{j+1}^k-U_{j-1}^k}{2h}=0 \tag{9.11}$$

is

$$v_t+av_x=-\frac{a^2\Delta t}{2}v_{xx}. \tag{9.12}$$

The highest derivative is similar to backward heat equation which is dynamically unstable!

9.6 Lax-Wendroff scheme.

We can add numerical viscosity to improve the unstable scheme to stable. Since

$$\begin{aligned}\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} &= u_t+\frac{\Delta t}{2}u_{tt}+O((\Delta t)^2)\\ &= u_t-\frac{1}{2}a^2(\Delta t)u_{xx}+O((\Delta t)^2).\end{aligned}$$

We can add the numerical viscosity to the $-\frac{1}{2}a^2\Delta u_{xx}$ to get the Lax-Wendroff scheme:

$$\frac{U_j^{k+1}-U_j^k}{\Delta t}+a\frac{U_{j+1}^k-U_{j-1}^k}{2h}=\frac{1}{2}\frac{a^2\Delta t}{h^2}\left(U_{j-1}^k-2U_j^k+U_{j+1}^k\right). \tag{9.13}$$

Lax-Wendroff scheme is second order accurate both in time and space. To show this, we investigate the local truncation error:

$$\begin{aligned}T(x,t) &= \frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}-\frac{a\left(u(x+h,t)-u(x-h,t)\right)}{2h}\\ &\quad -\frac{a^2\Delta t\left(u(x-h,t)-2u(x,t)+u(x+h,t)\right)}{2h^2}\\ &= u_t+\frac{\Delta t}{2}-au_{xx}-\frac{a^2\Delta t}{2}u_{xx}+O((\Delta t)^2+h^2)\\ &= O((\Delta t)^2+h^2)\end{aligned}$$

since $u_t = -au_x$ and $u_{tt} = -au_{xt} = -a\frac{\partial}{\partial x}u_t = a^2u_{xx}$.

The von Neumann stability analysis. The growth factor of Lax-Wendroff scheme is

$$\begin{aligned} g(\theta) &= 1 - \frac{\mu}{2}\left(e^{ih\xi} - e^{-ih\xi}\right) + \frac{\mu^2}{2}\left(e^{-ih\xi} - 2 + e^{ih\xi}\right) \\ &= 1 - \mu i \sin\theta - 2\mu^2 \sin^2(\theta/2), \end{aligned}$$

where again $\theta = h\xi$. Therefore we proceed with the following derivation

$$\begin{aligned} |g(\theta)|^2 &= \left(1 - 2\mu\sin^2\frac{\theta}{2}\right)^2 + \mu^2\sin^2\theta \\ &= 1 - 4\mu^2\sin^2\frac{\theta}{2} + 4\mu^4\sin^4\frac{\theta}{2} + 4\mu^2\sin^2\frac{\theta}{2}\left(1 - \sin^2\frac{\theta}{2}\right) \\ &= 1 - 4\mu^2\left(1 - \mu^2\right)\sin^4\frac{\theta}{2} \\ &\leq 1 - 4\mu^2\left(1 - \mu^2\right). \end{aligned}$$

We conclude that $|g(\theta)| \leq 1$ is $\mu \leq 1$, that is $\Delta t \leq h/|a|$. If $\Delta t > h/|a|$, there are ξ's such that $g(\theta)| > 1$ and the scheme is unstable.

The leading term of the modified PDE for the Lax-Wendroff method is

$$v_t + av_x = -\frac{1}{6}ah^2\left(1 - \left(\frac{a\Delta t}{h}\right)^2\right)v_{xxx} \tag{9.14}$$

This is a dispersive equation. The group velocity for the wave number ξ under Lax-Wendroff is

$$c_g = a - \frac{1}{2}ah^2\left(1 - \left(\frac{a\Delta t}{h}\right)^2\right)\xi^2 \tag{9.15}$$

which is less than a for all wave numbers. As a result the numerical result can be expected to develop a train of oscillations behind the peak, with the high wave numbers lagging farther behind the correct location, see Strikwerda. If we retain one more term in the modified eqaution for Lax-Wendroff, we will get:

$$v_t + av_x = \frac{1}{6}ah^2\left(\left(\frac{a\Delta t}{h}\right)^2 - 1\right)v_{xxx} - \epsilon v_{xxxx}, \tag{9.16}$$

where the ϵ is the fourth order dissipative term is $O(h^3)$ and positive when the stability bound holds. This high order dissipation causes the highest wave number to be damped, so that there is a limit to the oscillations.

9.7 Beam-Warming method.

Beam-Warming method is one sided finite difference scheme to the modified equation

$$v_t + av_x = \frac{a^2 \Delta t}{2} v_{xx}.$$

Recall the one sided finite difference formulas:

$$\begin{aligned} u'(x) &= \frac{3u(x) - 4u(x-h) + u(x-2h)}{2h} + O(h^2), \\ u''(x) &= \frac{u(x) - 2u(x-h) + u(x-2h)}{h^2} + O(h). \end{aligned}$$

The Beam-Warming method for $u_t + au_x = 0$ for $a > 0$ is

$$U_j^{k+1} = U_j^k - \frac{a\Delta t}{2h}\left(3U_j^k - 4U_{j-1}^k + U_{j-2}^k\right) + \frac{(a\Delta t)^2}{2h^2}\left(U_j^k - 2U_{j-1}^k + U_{j-2}^k\right) \tag{9.17}$$

The method is second order accurate in time and space if $\Delta t \sim h$. The CFL constraints is

$$0 < \Delta t \le \frac{2h}{a}. \tag{9.18}$$

For this method, we do not numerical boundary condition at $x = 1$, but we need a scheme to compute the solution U_1^j. The leading terms of the modified PDE for the Beam-Warming method is

$$v_t + av_x = \frac{1}{6} a h^2 \left(\left(\frac{a\Delta t}{h} \right)^2 - 1 \right) v_{xxx}. \tag{9.19}$$

In this case, the group velocity is greater than a for all wave numbers in the case $0 \le a\Delta t/h \le 1$, so that the oscillations move ahead of the main hump. If $1 \le a\Delta t/h \le 2$, then the group velocity is less than a and the oscillations fall behind.

9.8 The Crank-Nicholson scheme.

The Crank-Nicholson scheme for the advcetion equation $u_t + au_x = f$ is

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + a\frac{U_{j+1}^k - U_{j-1}^k + U_{j+1}^{k+1} - U_{j-1}^{k+1}}{4h}. = f_j^{k+\frac{1}{2}} \tag{9.20}$$

It is second order accurate in time and in space. It is also unconditionally stable. The method need a numerical boundary condition at $x = 1$. For one dimensional problem, it is very effective since it is very easy to solve tridiagonal system of equations. It may not be necessary for high dimensional problems since for hyperbolic equations, the time step constraint $\Delta t \sim h$ is not a major concern.

9.9 The method of lines (MOL).

Different MOL method can be used as well depending how the spatial derivative term is discretized. For the advection equation $u_t + au_x = 0$, if we use the following discretization

$$\frac{\partial U_i}{\partial t} + a\frac{U_{i+1} - U_{i-1}}{2h} = 0. \tag{9.21}$$

Then the ODE solver that is going to be used is likely to be implicit since the forward Euler method is unstable!

9.10 Numerical boundary conditions (NBC).

We need a numerical boundary condition at one end for the one-way wave equation when we use the Lax-Friedrics, Lax-Wendroff, Beam-Warming, and Leap-Frog schemes. There are several approaches that can be used

- Extrapolation. Recall the Lagrange interpolation formula

  $$f(x) \approx f(x_1)\frac{x - x_2}{x_1 - x_2} + f(x_2)\frac{x - x_1}{x_2 - x_1}$$

  We can use the same time level for the interpolation to get

  $$\begin{aligned} U_M^{k+1} &= U_{M-1}^{k+1}, \qquad \text{1-st order} \\ U_M^{k+1} &= U_{M-2}^{k+1}\frac{x_M - x_{M-1}}{x_{M-1} - x_M} + U_{M-1}^{k+1}\frac{x_M - x_{M-2}}{x_{M-2} - x_{M-1}} \qquad \text{2-nd order.} \end{aligned}$$

  If a uniform grid is used with spatial step size h, the formula above becomes

  $$U_M^{k+1} = -U_{M-2}^{k+1} + 2U_{M-1}^{k+1}.$$

- Quasi-characteristics. If we use previous time level for the interpolation, we get

  $$\begin{aligned} U_M^{k+1} &= U_{M-1}^{k}, \qquad \text{1-st order} \\ U_M^{k+1} &= U_{M-2}^{k}\frac{x_M - x_{M-1}}{x_{M-1} - x_M} + U_{M-1}^{k}\frac{x_M - x_{M-2}}{x_{M-2} - x_{M-1}} \qquad \text{2-nd order.} \end{aligned}$$

- Use the schemes that does not need NBC at or near the boundary, for example, the upwind scheme, the Beam-Warming method.

The accuracy, the stability of the numerical scheme usually depend on the numerical boundary conditions that is used. As a rule of thumb, the main scheme and the scheme for NBC both should be stable.

9.11 Second order linear hyperbolic PDEs.

Consider the wave equation:

$$
\begin{aligned}
& u_{tt} = a^2 u_{xx}, \qquad 0 \le x \le 1, \\
\text{IC:} \quad & u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \\
\text{BC:} \quad & u(0,t) = g_1(t), \quad u(1,t) = g_2(t).
\end{aligned}
$$

We can use D'Alembert's technique to find the exact solution if the domain is the entire space. Introduce:

$$
\left\{ \begin{array}{l} \xi = x - at \\ \eta = x + at, \end{array} \right. \qquad \text{or} \qquad \left\{ \begin{array}{l} x = \dfrac{\xi + \eta}{2} \\ t = \dfrac{\eta - \xi}{2}. \end{array} \right. \tag{9.22}
$$

Using the chain role, we get

$$
\begin{aligned}
u_t &= -a u_\xi + a u_\eta, \\
u_{tt} &= a^2 u_{\xi\xi} - 2a^2 u_{\xi\eta} + a^2 u_{\eta\eta}, \\
u_x &= u_\xi + u_\eta, \\
u_{xx} &= u_{\xi\xi} + u_{\xi\eta} + u_{\eta\eta}.
\end{aligned}
$$

Substitute these relations into the wave equation to get

$$
u_{\xi\xi} a^2 - 2a^2 u_{\xi\eta} + a^2 u_{\eta\eta} = a^2 \left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \right).
$$

The equation above then is simplified to

$$
4a^2 u_{\xi\eta} = 0.
$$

Therefore the solution can be obtained

$$
\begin{aligned}
& u_\xi = F(\xi), \quad \Longrightarrow \quad u(x,t) = F(\xi) + G(\eta), \\
& u(x,t) = F(x - at) + G(x + at).
\end{aligned}
$$

Particularly, if the domain is $(-\infty, \infty)$ then the solution is

$$
u(x,t) = \frac{1}{2} \Big(u(x - at, 0) + u(x + at, 0) \Big),
$$

which tells us that a signal (wave) will propogates along the characteristics $x - at$ and $x + at$ with speed a and half of the strength.

9.11.1 A finite difference method (CT-CT) for the second order wave equation.

$$\frac{U_j^{k+1} - 2U_j^k + U_j^{k-1}}{(\Delta t)^2} = a^2 \frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{h^2} \tag{9.23}$$

The method is second order accurate both in time and space $((\Delta t)^2 + h^2)$. The CFL constraints of this method is $\Delta t \le \frac{h}{|a|}$ this will be verified through the following discussion.

The von Neumann analysis gives:

$$\frac{g - 2 + 1/g}{(\Delta t)^2} = a^2 \frac{e^{-ih\xi} - 2 + e^{ih\xi}}{h^2}.$$

Let $\mu = \frac{|a|\Delta t}{h}$. The equation above becomes:

$$g^2 - 2g + 1 = \mu^2 \left(-4\sin^2\theta\right),$$

where $\theta = h\xi/2$. Solve the equation above to get

$$g = 1 \pm 2\mu|\sin\theta|\sqrt{g}i,$$

$$\text{or} \qquad g_{\pm} = \left(\sqrt{1 - a^2\mu^2\sin^2\theta} \pm i\mu a\sin\theta\right)^2$$

If $1 - a^2\mu^2\sin^2\theta \ge 0$ which is ture if Δt is small enough, we have

$$|g_{\pm}|^2 = \left|1 - a^2\mu^2\sin^2 + \mu^2a^2\sin^2\right|^2 = 1.$$

Note that the definition of stability for second order equations (in time) that contain u_{tt} is weaker than that of the first order equations that contain only u_t term.

A finite difference scheme for second order PDE (in time) $P_{\Delta t,h}v_j^k = 0$ is stable in a stability region Λ if there is an integer J such that for any positive time T, there is a constant C_T independent of Δt and h such that

$$\|\mathbf{v}^n\|_h \le \sqrt{1+n^2}\, C_T \sum_{j=0}^{J} \|\mathbf{v}^j\|_h \tag{9.24}$$

for any n that satisfies $0 \le n\Delta t \le T$ with $(\Delta t, h) \in \Lambda$.

The definition allows linear growth in time. Again, if a finite difference method is consistent and stable, then the finite difference method will converge.

9.11.2 Transform second order wave equation to a first order system.

While we can solve the second order wave equation directly, in this section, we discuss how to change the equation to a first order system. Most of discussions in the literature is about conservation laws which is a first order non-linear syste. A first order linear hyperbolic system has the form

$$\mathbf{u}_t = (A\mathbf{u})_x = A\mathbf{u}_x$$

which is a special case of one dimensional conservation laws

$$\mathbf{u}_t + (\mathbf{f}(\mathbf{u}))_x = 0, \qquad \mathbf{u}_t + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\mathbf{u}_x = 0.$$

For simplification of discussion, we set $a = 1$. Introduce

$$\begin{cases} p = u_t \\ q = u_x, \end{cases} \qquad u_{tt} = p_t, \quad q_x = u_{xx}.$$

Therefore we have

$$\begin{cases} p_t = u_{tt} = u_{xx} = q_x, \\ q_t = u_{xt} = (u_t)_x = p_x \end{cases}$$

In the matrix-vector form, it is

$$\begin{bmatrix} p \\ q \end{bmatrix}_t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}_x. \tag{9.25}$$

The eigenvalues of A is -1 and 1, therefore the system is hyperbolic.

9.11.3 Initial and boundary conditions for the system.

From the given boundary conditions we get

$$u(0,t) = g_1(t), \qquad u_t(0,t) = g_1'(t) = p(0,t),$$
$$u(1,t) = g_2(t), \qquad u_t(0,t) = g_1'(t) = p(1,t).$$

There is no boundary condition for $q(x,t)$.

Now consider the initial conditions

$$p(x,0) = u_t(x,0) = u_1(x), \quad \text{known},$$
$$q(x,0) = u_x(x,0) = \frac{\partial}{\partial x}u(x,0) = u_0'(x), \quad \text{known}.$$

To solve a hyperbolic system numerically, usually we change the system to a diagonal form (characteristic direction) so that we can determine the boundary conditions and appropriate numerical methods (such as up-wind method). Let $A = T^{-1}DT$, where

$D = diag(\lambda_1, \lambda_2, \cdots \lambda_n)$ is a diagonal matrix containing eigenvalues of A, T is a non-singular matrix. Then we have

$$\mathbf{u}_t = A\mathbf{u}_x, \qquad T\mathbf{u}_t = TAT^{-1}T\mathbf{u}_x, \qquad (T\mathbf{u})_t = D\,(T\mathbf{u})_x\,.$$

Let $\tilde{\mathbf{u}} = T\mathbf{u}$. We get a new first order system

$$\tilde{\mathbf{u}}_t = D\tilde{\mathbf{u}}_x,$$

or $(\tilde{u}_i)_t = \lambda_i(\tilde{u}_i)_x$, $i = 1, 2, \cdots, n$ which we know how do solve them one by one. We also know at which end that we should have a boundary condition depending on the sign of λ_i.

For the second order wave equation, we know the eigenvalue is 1 and -1. The unit eigenvector corresponding to the eigenvalue 1 can be found by solving $Ax = x$, $\|x\|_2 = 1$. we can get $x = [1,\ 1]^T/\sqrt{2}$. Similarly the unit eigenvector corresponding to the eigenvalue -1 is $x = [-1,\ 1]^T/\sqrt{2}$. Therefore we have

$$T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The transform can be written below:

$$\begin{aligned} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}_t &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}_x \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}_x. \end{aligned}$$

In component form, the first order system above is

$$\begin{aligned} \left(\frac{1}{\sqrt{2}}p - \frac{1}{\sqrt{2}}q\right)_t &= -\left(\frac{1}{\sqrt{2}}p - \frac{1}{\sqrt{2}}q\right)_x \\ \left(\frac{1}{\sqrt{2}}p + \frac{1}{\sqrt{2}}q\right)_t &= \left(\frac{1}{\sqrt{2}}p + \frac{1}{\sqrt{2}}q\right)_x. \end{aligned}$$

Let

$$\begin{cases} y_1 = \dfrac{1}{\sqrt{2}}p - \dfrac{1}{\sqrt{2}}q, \\ y_2 = \dfrac{1}{\sqrt{2}}p + \dfrac{1}{\sqrt{2}}q. \end{cases}$$

We get:

$$\begin{cases} \dfrac{\partial}{\partial t} y_1 = -\dfrac{\partial}{\partial x} y_1, \\ \dfrac{\partial}{\partial t} y_2 = -\dfrac{\partial}{\partial x} y_2. \end{cases}$$

Now we have two separate one-way wave equations and we can use various numerical methods. We know the intial conditions. We need a boundary condition for y_1 at $x = 1$ and a boundary condition for y_2 at $x = 0$. Note that

$$\begin{aligned} y_1(0,t) &= \frac{1}{\sqrt{2}} p(0,t) - \frac{1}{\sqrt{2}} q(0,t), \\ y_2(0,t) &= \frac{1}{\sqrt{2}} p(0,t) + \frac{1}{\sqrt{2}} q(0,t) \end{aligned}$$

and $q(0,t)$ is unknown. However,

$$y_1(0,t) + y_2(0,t) = \frac{2}{\sqrt{2}} p(0,t)$$

is known. We can use the followin steps to determine the boundary condition at $x = 0$.

1. Update $(y_1)_0^{k+1}$ first which we do not need a boundary condition.
2. Use $(y_2)_0^{k+1} = \frac{2}{\sqrt{2}} p_0^{k+1} - (y_1)_0^{k+1}$.

Similar method can be applied at $x = 1$.

9.12 Some commonly used finite difference methods for a linear system $\mathbf{u}_t + A\mathbf{u}_x = 0$.

- Backward Euler method:

$$\mathbf{U}_j^{k+1} = \mathbf{U}_j^k - \frac{\Delta t}{2h} A \left(\mathbf{U}_{j+1}^{k+1} - \mathbf{U}_{j-1}^{k+1} \right). \tag{9.26}$$

  Note that fackward Euler method does not work.

- Lax-Friedrichs scheme

$$\mathbf{U}_j^{k+1} = \frac{1}{2} \left(\mathbf{U}_{j+1}^k + \mathbf{U}_{j-1}^k \right) - \frac{\Delta t}{2h} A \left(\mathbf{U}_{j+1}^k - \mathbf{U}_{j-1}^k \right). \tag{9.27}$$

- Leap-Frog scheme

$$\mathbf{U}_j^{k+1} = \mathbf{U}_j^{k-1} - \frac{\Delta t}{2h} A \left(\mathbf{U}_{j+1}^k - \mathbf{U}_{j-1}^k \right). \tag{9.28}$$

- Lax-Wendroff scheme

$$\mathbf{U}_j^{k+1} = \mathbf{U}_j^k - \frac{\Delta t}{2h} A \left(\mathbf{U}_{j+1}^k - \mathbf{U}_{j-1}^k \right) + \frac{(\Delta t)^2}{2h^2} A^2 \left(\mathbf{U}_{j-1}^k - 2\mathbf{U}_j^k + \mathbf{U}_{j+1}^k \right). \tag{9.29}$$

To determine correct boundary conditions, usually we need to find the diagonal form $A = T^{-1} D T$ and the new system $\tilde{\mathbf{u}}_t = D \tilde{\mathbf{u}}_x$ with $\tilde{\mathbf{u}} = T\mathbf{u}$.

9.13 Finite difference methods for conservation laws.

The canonic form for one dimensional conservation law is

$$\mathbf{u}_t + \mathbf{f}(u)_x = 0. \tag{9.30}$$

One famous benchmark problem is the Burge's equation (scalar)

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{9.31}$$

in which $f(u) = u^2/2$. $\mathbf{f}(u)$ is often called the flux. Note that the Burge's equation can be written as a non-conservative form

$$u_t + uu_x = 0. \tag{9.32}$$

For the Burge's equation, the solution likely form shock(s) where the solution is discontinuous[3] even if the initial condition is arbitrarily differentiable, for example, $u_0(x) = \sin x$.

We can use the pwind scheme to solve the Burge's equation, if we use the con-conservative form, we have

$$\begin{aligned}
\frac{U_j^{k+1} - U_j^k}{\Delta t} + U_j^k \frac{U_j^k - U_{j-1}^k}{h} &= 0, \quad \text{if } U_j^k \geq 0, \\
\frac{U_j^{k+1} - U_j^k}{\Delta t} + U_j^k \frac{U_{j+1}^k - U_j^k}{h} &= 0, \quad \text{if } U_j^k < 0,
\end{aligned}$$

or the conservative form

$$\begin{aligned}
\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{(U_j^k)^2 - (U_{j-1}^k)^2}{2h} &= 0, \quad \text{if } U_j^k \geq 0, \\
\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{(U_{j+1}^k)^2 - (U_j^k)^2}{2h} &= 0, \quad \text{if } U_j^k < 0.
\end{aligned}$$

If the solution is smooth, both methods work well (first order accurate). However, the conservative form gives much better results than the non-conservative form if shocks develop.

We can derive the Lax-Wendroff scheme using the modifed equation of the non-conservative form. Since $u_t = -uu_x$, we have

$$\begin{aligned}
u_{tt} &= -u_t u_x - u u_{tx} \\
&= uu_x^2 + u(uu_x)_x \\
&= uu_x^2 + u\left(u_x^2 + uu_{xx}\right) \\
&= 2uu_x^2 + u^2 u_{xx}.
\end{aligned}$$

[3]There is no classical solution to the PDE when shocks develop because u_x is not well defined. We need to look for weak solutions.

Thus the leading terms of the modified equation for the first order method is

$$u_t + uu_x = \frac{\Delta t}{2}\left(2uu_x^2 + u^2 u_{xx}\right). \tag{9.33}$$

Therefore the non-conservative Lax-Wendroff scheme for the Burge's equation is

$$\begin{aligned} U_j^{k+1} &= U_j^k - \Delta t U_j^k \frac{U_{j+1}^k - U_{j-1}^k}{2h} \\ &= + \frac{(\Delta t)^2}{2}\left(2U_j^k\left(\frac{U_{j+1}^k - U_{j-1}^k}{2h}\right)^2 + (U_j^k)^2 \frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{h^2}\right) \end{aligned}$$

9.14 Conservative finite difference method for conservation laws

Consider

$$\mathbf{u}_t + \mathbf{f}(u)_x = 0.$$

We seek a numerical scheme of the form:

$$\mathbf{u}_j^{k+1} = \mathbf{u}_j^k - \frac{\Delta t}{h}\left(\mathbf{g}_{j+\frac{1}{2}}^k - \mathbf{g}_{j-\frac{1}{2}}^k\right) \tag{9.34}$$

where

$$\mathbf{g}_{j+\frac{1}{2}} = \mathbf{g}\left(\mathbf{u}_{j-p+1}^k, \mathbf{u}_{j-p+2}^k, \cdots, \mathbf{u}_{j+q+1}^k\right)$$

is called the numerical flux and satisfies

$$g(u, u, \cdots, u) = f(u). \tag{9.35}$$

Such a scheme is called conservative scheme. For example, we can take $g(u) = u^2/2$ for the burge's equation.

Below we derive the general criteria that g should satisfy.

1. Integrate the equation with respect to x from $x_{j-\frac{1}{2}}$ to $x_{j+\frac{1}{2}}$, we get

$$\begin{aligned} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t dx &= -\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(u)_x dx \\ &= -\left(f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t))\right) \end{aligned}$$

2. Integrate the equation above with respect to t from t^k to t^{k+1}, we get

$$\int_{t^k}^{t^{k+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t dx dt = -\left\{\int_{t^k}^{t^{k+1}} \left(f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t))\right) dt\right\},$$

$$\int_{t^k}^{t^{k+1}} \left(u(x, t^{k+1}) - u(x, t^k)\right) dx = -\left\{\int_{t^k}^{t^{k+1}} \left(f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t))\right) dt\right\}.$$

Let

$$\bar{u}_j^k = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t^k)dx \tag{9.36}$$

which is called the cell average of $u(x,t)$ between the cell $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ at time level k. Then the expression that we derived earlier can be written as

$$\begin{aligned}
\bar{u}_j^{k+1} &= \bar{u}_j^k - \frac{1}{h}\left(\int_{t^k}^{t^{k+1}} f(u(x_{j+\frac{1}{2}},t))dt - \int_{t^k}^{t^{k+1}} f(u(x_{j-\frac{1}{2}},t))dt\right) \\
&= \bar{u}_j^k - \frac{\Delta t}{h}\left(\frac{1}{\Delta t}\int_{t^k}^{t^{k+1}} f(u(x_{j+\frac{1}{2}},t))dt - \frac{1}{\Delta t}\int_{t^k}^{t^{k+1}} f(u(x_{j-\frac{1}{2}},t))dt\right) \\
&= \bar{u}_j^k - \frac{\Delta t}{h}\left(g_{j+\frac{1}{2}} - g_{j+\frac{1}{2}}\right),
\end{aligned}$$

where

$$g_{j+\frac{1}{2}} = \frac{1}{\Delta t}\int_{t^k}^{t^{k+1}} f(u(x_{j+\frac{1}{2}},t))dt.$$

Different conservative schemes can be obtained if different approximations are used to evaluate the integral above.

9.15 Some commonly used numerical scheme for conservation laws.

- Lax-Friedrichs scheme

$$U_j^{k+1} = \frac{1}{2}\left(U_{j+1}^k + U_{j-1}^k\right) - \frac{\Delta t}{2h}\left(f(U_{j+1}^k) - f(U_{j-1}^k)\right). \tag{9.37}$$

- Lax-Wendroff scheme:

$$\begin{aligned}
U_j^{k+1} &= U_j^k - \frac{\Delta t}{2h}\left(f(U_{j+1}^k) - f(U_{j-1}^k)\right) \\
&\quad + \frac{(\Delta t)^2}{2h^2}\left\{A_{j+\frac{1}{2}}\left(f(U_{j+1}^k) - f(U_j^k)\right) - A_{j-\frac{1}{2}}\left(f(U_j^k) - f(U_{j-1}^k)\right)\right\}.
\end{aligned} \tag{9.38}$$

where $A_{j+\frac{1}{2}} = Df(u(x_{j+\frac{1}{2}},t))$ is the Jacobian matrix of $f(u)$ at $u(x_{j+\frac{1}{2}},t))$.

A modified version called Lax-Wendroff-Richtmyer scheme which does not need to the Jacobian matrix is listed below:

$$\left\{\begin{array}{l}
U_{j+\frac{1}{2}}^{k+\frac{1}{2}} = \frac{1}{2}\left(U_j^k + U_{j+1}^k\right) - \frac{\Delta t}{2h}\left(f(U_{j+1}^k) - f(U_j^k)\right) \\
U_j^{k+1} = U_j^k - \frac{\Delta t}{h}\left(f(U_{j+\frac{1}{2}}^{k+\frac{1}{2}}) - f(U_{j-\frac{1}{2}}^{k+\frac{1}{2}})\right).
\end{array}\right. \tag{9.39}$$