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Review paper

A brief overview of non-overlapping domain decomposition methods

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Resumen

Se presenta una visión general de los métodos de descomposición de dominio con dominios ajenos. Los métodos más eficientes que existen en la actualidad, el BDDC y el FETI-DP, se ubican en un marco 'primal' (el 'espacio de vectores derivados (DVS, por sus siglas en inglés)'), el cual permite una presentación sintética y efectiva tanto de las formulaciones primales como de las 'duales'. El marco conceptual del espacio de los vectores derivados tiene alguna similitud con el que usa BDDC, pero una diferencia importante es que en el marco DVS el problema tratado se transforma en otro definido en el espacio vectorial producto, mientras que en el BDDC no se hace tal cosa. Esto simplifica los algoritmos, los cual se sintetizan en un breve conjunto de fórmulas matriciales muy generales que son aplicables a matrices simétricas, no simétricas e indefinidas, cuando ellas provienen de la discretización de ecuaciones diferenciales parciales o sistemas de tales ecuaciones. Las fórmulas matriciales de este conjunto, son explícitas y pueden ser usadas directamente para desarrollar códigos computacionales. Hasta donde sabemos, dos de los algoritmos precondicionados del conjunto mencionado, son totalmente diferentes a cualquiera de los reportados en la literatura y deben ser motivo de investigaciones futuras.

Palabras clave: subestructuración iterativa, métodos de descomposición en dominios ajenos; BDD, BDDC; FETI, FETI-DP; pre condicionadores; espacio producto; multiplicadores de Lagrange.

Abstract

An overview of non-overlapping domain decomposition methods is presented. The most efficient methods that exist at present, BDDC and FETI-DP, are placed in a 'primal' framework (the 'derived-vectors space (DVS)') which permits a synthetic and effective presentation of both: primal and 'dual' formulations. The derived-vectors space is similar to the setting used in BDDC. A significant difference is that, in the DVS framework, the problem considered is transformed into one that is defined in a product vector space while in BDDC that is not done. This simplifies the algorithmic formulations, which are summarized in a set of matrix-formulas applicable to symmetric, non-symmetric and indefinite matrices generated when treating numerically partial differential equations or systems of such equations. They can directly be used for code development. Two preconditioned algorithms of the mentioned set had not been reported previously in the DDM literature, as far as we know, and are suitable for being researched.

Key words: iterative substructuring, non-overlapping domain decomposition, BDD, BDDC, FETI, FETI-DP, preconditioners, product space, Lagrange multipliers.

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Introduction

In this paper we present a synthetic and brief overview of some of the most important algebraic formulas of non-overlapping domain decomposition methods (DDM). We will use a framework that is very convenient for this purpose, which will be called the *'derived-vector space (DVS)'* framework [Herrera and Yates 2010; Herrera and Yates 2009].

Among the frameworks that are used in non-overlapping DDM formulations two categories are distinguished: *dual* frameworks, which as in the case of FETI and its variants use Lagrange multipliers, and *primal* frameworks, which as in the case of BDD and its variants tackle the problems directly without resource to Lagrange multipliers [Dohrmann 2003; Mandel and Dohrmann 2003; Mandel *et al*., 2005; Toselli and Widlund 2005]. The *derived-vector space (DVS)* framework used here is a *primal* framework similar to that of the BDDC formulations. A significant difference between the *DVS* framework and that of BDDC formulations is that in the *DVS-framework* the problem is transformed into one defined in the *derived-vectors space*, which is a product space containing the discontinuous functions, and thereafter all the work is carried out in it. In BDDC formulations, on the other hand, the original space of continuous functions is never completely abandoned; indeed, one frequently goes back and forth from the degrees of freedom associated with the original space of continuous functions to the degrees of freedom associated with the substructures, which in such formulations play the role of the product space (see Section 15 for a more detailed discussion).

Although the *DVS* framework is a *primal* framework, *dual* formulations can also be accommodated in it; this feature permits unifying in its realm both *dual* and *primal* formulations; in particular, BDDC and FETI-DP. Also, the *derived-vectors space* constitutes a Hilbert-space with respect to a suitable inner product -the *Euclidean inner-product*- and, while using the *DVS* formulation, we will profit from its Hilbert-space structure achieving in this manner great simplicity for the algorithm formulations. Furthermore, the *theory of partial differential equations in discontinuous piecewise defined functions* [Herrera 2007] is used for establishing clear correspondences between the problems at the continuous level and those obtained after discretization (see, Section 9 of [Herrera and Yates 2010] and Appendix "B" of the present article). In particular, in this paper using the *DVS-framework* we present simple explicit matrix formulas that can be applied to simplify code development of models governed by a single differential equation or systems of such equations; they have a wide range of applications to practical problems which includes non-symmetric and indefinite matrices. We also remark that all our developments are carried out in vector spaces subjected to constraints and therefore all the *DVS* algorithms here presented are algorithms with constraints.

In this paper, a survey of the non-overlapping DDM algorithms that can be developed in the *DVS framework* is carried out, which yields a synthetic and brief overview of some of the most important algebraic formulas of non-overlapping domain decomposition methods (DDM). In particular, FETI-DP [Farhat and Roux 1991; Mandel and Tezaur 1996; Farhat *et al*., 2001; Toselli and Widlund 2005] and the BDDC [Dohrmann 2003; Mandel and Dohrmann 2003; Mandel *et al*., 2005; Mandel 1993; Mandel and Brezina 1993; Mandel and Tezaur 2001], which are the most successful nonoverlapping DDM, are incorporated producing in this manner *DVS-versions* of them. In recent years a number of papers have discussed connections between BDDC and FETI-DP [Mandel *et al*., 2005; Li and Widlund 2006; Klawonn and Widlund 2001], and similar connections encountered using the *DVS-framework* are here discussed. Also, by now in the literature the developments on DMM for non-symmetric and indefinite matrices have been significant (see for example [Da Conceição 2006; Farhat and Li 2005; Li and Tu 2009; Toselli 2000; Tu and Li; Tu and Li]). As said before, the *DVS-framework* for non-overlapping DDM is applicable to such kind of matrices; indeed, [Herrera and Yates 2009] was devoted to extend the *DVS framework* to non-symmetric and indefinite matrices. The assumptions under which such extension is possible were spelled out with precision and detail in Section 9 of [Herrera and Yates 2009]. When such results are complemented with those presented in Sections 7 to 14 of this paper, they permit establishing with certainty and precision in each case when such algorithms can be applied. Thus far, we have not seen discussed this topic with this generality elsewhere, in spite of its obvious importance.

In conclusion, our results can be effectively summarized in eight matrix-formulas; of them, those with greater practical interest are of course the preconditioned formulations. The non-preconditioned ones are included because they are important for understanding properly the theoretical developments. Of the four preconditioned matrix formulas contained in that summary, as said before, two correspond to the BDDC and FETI-DP algorithms, while for the other two we have not been able to find suitable counterparts in the DDM literature already published, although the effectiveness of their performance can be expected to be of the same order as BDDC or FETI-DP.

The paper is organized as follows. Section 2 is devoted to present an overview of the *DVS framework*. The problem to be dealt with (the *original problem*) is stated in Section3, while Sections 4 and 5 introduce the notions of the *derived-vectors space*. The general problem with constraints defined in the *derived-vector space*, equivalent to the *original-problem*, is formulated in Section 6. The guidelines for the manner in which Sections 7 to 14 were organized is supplied by the results of Appendix "B". Only two basic *non-preconditioned* algorithms were considered: the Dirichlet-Dirichlet and the Neumann-Neumann algorithms; a *primal* and a *dual* formulation is supplied for each one of them. In turn, each one of the four non-preconditioned formulations so obtained is preconditioned. This yields, in total, the eight algorithms mentioned before. Sections 7 to 10 are devoted to the four non-preconditioned algorithms, while Sections 11 to 14 are devoted to the preconditioned algorithms. The *DVS* versions of BDDC and FETI-DP are obtained when the *primal* and *dual* formulations of the non-preconditioned Dirichlet-Dirichlet algorithms are pre-conditioned, respectively. Some comparisons and comments about FETI-DP and BDDC as seen from the *DVS framework* are made in Section 15, while the Conclusions are presented in Section 16. Three Appendices are included in which some complementary technical details are given.

Section 2
Overview of the DVS framework

In previous papers [Herrera and Yates 2010; Herrera and Yates 2009] a general framework for domain decomposition methods, here called the *'derived vector space framework (DVS-framework)'*, has been developed. Its formulation starts with the system of linear equations that is obtained after the partial differential equation, or system of such equations, has been discretized. We shall call, this system of linear equations, the *'original problem'*. Independently of the discretization method used, it is assumed that a set of nodes and a domain-partition have been defined and that both the nodes and the partition-subdomains have been numbered. Generally, some nodes belong to more than one partition-subdomain (Figure 1). For the formulation of non-overlapping domain decomposition methods, this is an inconvenient feature. To overcome this problem, the *DVS framework* introduces a new set of nodes, the *'derived nodes'*; a *derived node* is a pair of natural numbers: a *node-index* followed by a *subdomain-index*, which may be any that fulfills the condition that the node involved belongs to the corresponding partition-subdomain. As for the node-indices, they are referred to as the *'original-nodes'*.

Furthermore, with each partition-subdomain we associate a *'local subset of derived-nodes'*, which is constituted by the *derived-nodes* whose *subdomain-index* corresponds to that partition-subdomain. The family of *local subsets of derived-nodes* so obtained, one for each partition-subdomain, constitutes a truly disjoint (i.e., non-overlapping) partition of the whole set of *derived-nodes* (Figure 2). Therefore, it is adequate for overcoming the difficulty mentioned above. Thereafter, the developments are relatively straightforward. A *'derived-vector'* is defined to be a real-valued function[1] defined in the whole set of *derived-nodes*; the set of all *derived-vectors* constitutes a linear space: the *'derived-vector space (DVS)'*. This latter vector-space must be distinguished from that constituted by the real-valued functions defined in the *original-nodes*, which is referred to as the *'original-vector space'*. A new problem, which is equivalent to the *original problem*, is defined in the *derived-vector space*. Of course, the matrix

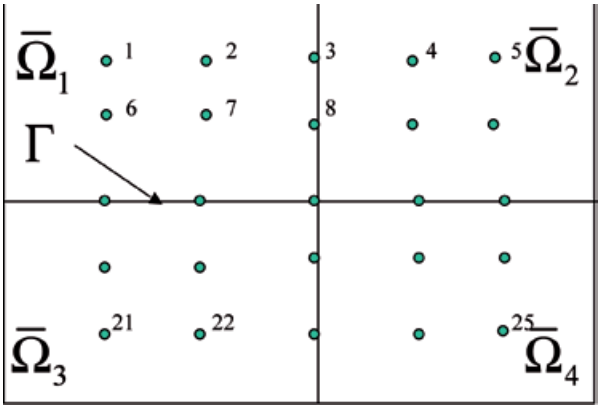

Figure 1. The "original nodes".

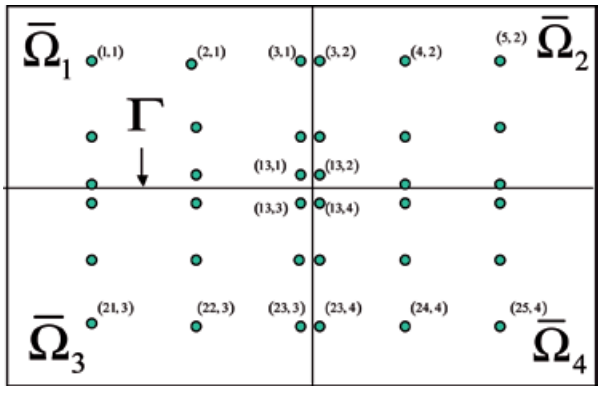

Figure 2. The "derived nodes".

[1] For the treatment of systems of equations, vector-valued functions are considered, instead

of this new problem is different to the *original-matrix*, which is not defined in the *derived-vector space*, and the theory supplies a formula for deriving it; the procedure for constructing it is similar to *substructuring* (see, Appendix "A"). From there on, in the *DVS framework*, all the work is done in the *derived-vector space* and one never goes back to the *original vector-space*.

The *derived-vector space* is a kind of product-space; namely, the product of the family of *local subsets of derived-nodes* mentioned above. Fur-thermore, it constitutes a Hilbert-space (finite-dimensional) with respect to a suitable inner-product, called the Euclidean inner product, and it is handled as such throughout. Although the *DVS framework* was originally developed having in mind applications to symmetric and definite matrices, in [Herrera and Yates 2009] it was extended to nonsymmetric and indefinite matrices. The assumptions under which such extensions are possible were spelled out in detail there (see Section 9 of [Herrera and Yates 2009]). In this paper, we carry out a survey, as exhaustive as possible, of the DDM algorithms that can be developed in the *DVS framework*, both preconditioned and non-preconditioned. Thereby, *DVS versions* of both the *BDDC* and *FETI-DP* algorithms are produced.

At the continuous level, the most studied procedures are the Neumann-Neuman and the Dirichlet-Dirichlet algorithms [Toselli and Widlund 2005; Quarteroni and Valli 1999]. During the development of the *DVS framework*, very precise and clear correspondences between the processes at the continuous level, before discretization, and the processes at the discrete level, after discretization, were established [Herrera and Rubio 2011]. Using such correspondences the results of our survey can be summarized in a brief and effective manner. They are:

I*) Non-Preconditioned Algorithms*

a) The *primal* Dirichlet-Dirichlet problem (Schur-complement algorithm)
b) The *dual* formulation of the Neumann-Neumann problem
c) The *primal* formulation of the Neumann-Neumann problem
d) The second *dual* formulation of the Neumann-Neumann problem

II*) Preconditioned Algorithms*

a) Preconditioned Dirichlet-Dirichlet (The *DVS-version of BDDC*)
b) Preconditioned *dual* formulation of the Neumann-Neumann problem (The *DVS-version of FETI-DP*)
c) Preconditioned *primal* formulation of the Neumann-Neumann problem
d) Preconditioned second *dual* formulation of Neumann-Neumann problem

All these algorithms are formulated in vector spaces subjected to constraints, so the algorithms are *constrained* algorithms.

Section 3
The original problem

The *DVS-framework* applies to the system-matrix that is obtained after discretization. Its procedures are independent of the discretization method used; it could be, FEM, finite-differences, or any other. It requires, however, that some assumptions (or axioms) be fulfilled, as it is explained in what follows. Such axioms are stated in terms of the system-matrix and two additional concepts: the *original nodes* and a family of subsets of such nodes, which is associated with a *domain partition* (or, *domain decomposition*). To illustrate how such concepts are introduced, consider a variational formulation of the discretized version of a general boundary value problem. It consists in finding $\hat{u}\in V$, such that

$$\tilde{a}(\hat{u},v) = (\mathrm{g}, v), \forall v\in V \qquad (3.1)$$

Here, V is a finite dimensional linear space of real-valued[2] functions defined in certain spatial domain Ω, while $g\in V$ is a given function.

Let $\hat{\mathrm{N}}\equiv\{1, \ldots, n\}$ be the set of indices, which number the nodes used in the discretization, and $\{\varphi_1, \ldots, \varphi_n\}\subset V$ be a basis of V, such that for each $i\in\hat{\mathrm{N}}$, $\varphi_i=1$ at node i and zero at every other node. Then, since $\hat{u}\in V$, we have:

$$\hat{u} = \sum_{i=1}^{n} \hat{u}_i \varphi_i \qquad (3.2)$$

Here, $\hat{u}_i$ is the value of $\hat{u}$ at node i. Let $\underline{\hat{u}}$ and $\underline{\hat{f}}$ be the vectors $\underline{\hat{u}}\equiv(\hat{u}_1, \ldots, \hat{u}_n)$ and $\underline{\hat{f}}\equiv\left(\hat{f}_1,\ldots,\hat{f}_n\right)$[3], with

$$\hat{f}_i \equiv (g, \varphi_i), i\in\hat{\mathrm{N}} \qquad (3.3)$$

The variational formulation of Eq. (3.1) is equivalent to:

$$\underline{\underline{\hat{A}}}\,\underline{\hat{u}} = \underline{\hat{f}} \qquad (3.4)$$

[2] The theory to be presented, with slight modifications, works as well in the case that the functions of V are vector-valued.
[3] Strictly, these should be column-vectors. However, when they are incorporated in the middle of the text, we write them as row-vectors to save space.

The matrix $\underline{\underline{\hat{A}}}$, which will be referred to as the *'original matrix'*, is given by

$$\underline{\underline{\hat{A}}} \equiv (\hat{A}_{ij}) \; with \; \underline{\underline{\hat{A}}}_{ij} \equiv \tilde{a}(\varphi_j, \varphi_i), i, j = 1, ..., n \tag{3.5}$$

After the problem has been discretized, a partition of Ω into a set of non-overlapping subdomains, $\{\Omega_1, ..., \Omega_E\}$, is introduced; more precisely, for each $\alpha = 1, ..., E$, Ω_α, is open and:

$$\Omega_\alpha \cap \Omega_\beta = \varnothing \; and \; \Omega \subset \bigcup_{\alpha=1}^{E} \bar{\Omega}_\alpha \tag{3.6}$$

Where $\bar{\Omega}_\alpha$ stands for the closure of Ω_α. The set of *'subdomain-indices'* will be

$$\hat{E} \equiv \{1, ..., E\} \tag{3.7}$$

$\hat{\mathrm{N}}^\alpha$, $\alpha = 1, ..., E$, will be used for the subset of *original-nodes* that correspond to nodes pertaining to $\bar{\Omega}_\alpha$. As usual, nodes will be classified into *'internal'* and *'interface-nodes'*: a node is *internal* if it belongs to only one partition-subdomain closure and it is an *interface-node*, when it belongs to more than one. For the application of *dual-primal* methods, *interface-nodes* are classified into *'primal'* and *'dual'* nodes. We define:

$\hat{\mathrm{N}}_\mathrm{I} \subset \hat{\mathrm{N}}$ as the set of *internal-nodes*;
$\hat{\mathrm{N}}_\Gamma \subset \hat{\mathrm{N}}$ as the set of *interface-nodes*;
$\hat{\mathrm{N}}_\pi \subset \hat{\mathrm{N}}$ as the set of *primal-nodes*[4]; and
$\hat{\mathrm{N}}_\Delta \subset \hat{\mathrm{N}}$ as the set of *dual-nodes*.

The set $\hat{\mathrm{N}}_\pi \subset \hat{\mathrm{N}}_\Gamma$ is chosen arbitrarily and then $\hat{\mathrm{N}}_\Delta$ is defined as $\hat{\mathrm{N}}_\Delta \equiv \hat{\mathrm{N}}_\Gamma - \hat{\mathrm{N}}_\pi$. Each one of the following two families of node-subsets is disjoint: $\{\hat{\mathrm{N}}_\mathrm{I}, \hat{\mathrm{N}}_\Gamma\}$ and $\{\hat{\mathrm{N}}_\mathrm{I}, \hat{\mathrm{N}}_\pi, \hat{\mathrm{N}}_\Delta\}$. Furthermore, these node subsets fulfill the relations:

$$\hat{\mathrm{N}} = \hat{\mathrm{N}}_\mathrm{I} \cup \hat{\mathrm{N}}_\Gamma = \hat{\mathrm{N}}_\mathrm{I} \cup \hat{\mathrm{N}}_\pi \cup \hat{\mathrm{N}}_\Delta \; and \; \hat{\mathrm{N}}_\Gamma = \hat{\mathrm{N}}_\pi \cup \hat{\mathrm{N}}_\Delta \tag{3.8}$$

The real-valued functions defined in $\hat{\mathrm{N}} = \{1, ..., n\}$ constitute a linear vector space that will be denoted by $\widehat{W}$ and referred to as the *'original vector-space'*. Vectors $\underline{\hat{u}} \in \widehat{W}$ will be written as $\underline{\hat{u}} = (\hat{u}_1, ..., \hat{u}_n)$, where $\hat{u}_i$ for $i = 1, ..., n$, are the *components* of the vector $\underline{\hat{u}}$. Then, by the *'original-problem'* consists in: *"Given $\underline{\hat{f}} \in \widehat{W}$, find a $\underline{\hat{u}} \in \widehat{W}$ such that Eq. (3.4) is fulfilled"*. Throughout our developments the *original matrix* $\underline{\underline{\hat{A}}}$ is assumed to be non-singular (i.e., it defines a bijection of $\widehat{W}$ into itself).

Conditions under which the DVS-framework is applicable to indefinite or/and non-symmetric were given in [Herrera and Yates 2009]; in particular, the following assumption (*'axiom'*) is adopted here: "Let the indices $i \in \hat{\mathrm{N}}^\alpha$ and $j \in \hat{\mathrm{N}}^\beta$ be *internal original-nodes*, then:

$$\hat{A}_{ij} = 0, \; whenever \; \alpha \neq \beta \tag{3.9}"$$

Section 4
Derived-Nodes

As said before, when a non-overlapping partition is introduced some of the nodes used in the discretization belong to more than one partition-subdomain. To overcome this inconvenient feature in the DVS-framework, besides the *original-nodes*, another set of nodes is introduced, called the *'derived nodes'*. The general developments are better understood, through a simple example that we explain first.

Consider the set of twenty five nodes of a "non-overlapping" domain decomposition, which consists of four subdomains, as shown in Figure 1. Thus, we have a set of nodes and a set of subdomains, which are numbered using of the index-sets: $\hat{\mathrm{N}} \equiv \{1, ..., 25\}$ and $\hat{E} \equiv \{1, 2, 3, 4\}$, respectively. Then, the sets of nodes corresponding to such a non-overlapping domain decomposition is actually overlapping, since the four subsets

$$\begin{aligned} \hat{\mathrm{N}}^1 &\equiv \{1, 2, 3, 6, 7, 8, 11, 12, 13\} \\ \hat{\mathrm{N}}^2 &\equiv \{3, 4, 5, 8, 9, 10, 13, 14, 15\} \\ \hat{\mathrm{N}}^3 &\equiv \{11, 12, 13, 16, 17, 18, 21, 22, 23\} \\ \hat{\mathrm{N}}^4 &\equiv \{13, 14, 15, 18, 19, 20, 23, 24, 25\} \end{aligned} \tag{4.1}$$

are not disjoint (see, Figure 1). Indeed, for example:

$$\hat{\mathrm{N}}^1 \cap \hat{\mathrm{N}}^2 = \{3, 8, 13\} \tag{4.2}$$

In order to obtain a "truly" non-overlapping decomposition, we replace the set of *'original nodes'* by another set: the set of *'derived nodes'*; a *'derived node'* is defined to be a pair of numbers: (p, α), where p corresponds a node that belongs to $\bar{\Omega}_\alpha$. In symbols: a *'derived node'* is a pair of numbers (p, α) such that $p \in \hat{\mathrm{N}}^\alpha$. We denote by X the set of *derived nodes*; we observe that the total number of *derived-nodes* is 36 while that of *original-nodes* is 25. Then, we define X^α as the set of *derived nodes* that can be written as (p, α), where α is kept fixed. Taking α successively as 1, 2, 3 and 4, we obtain the family of four subsets, $\{\mathrm{X}^1, \mathrm{X}^2, \mathrm{X}^3, \mathrm{X}^4\}$, which is a truly disjoint

[4]In order to mimic standard notations, as we try to do in most of this paper, we should use Π instead of the *low-case* π. However, we have found convenient to reserve the letter Π for another use.

decomposition of X, in the sense that (see, Figure 2):

$$\mathrm{X}=\bigcup_{\alpha=1}^{4}\mathrm{X}^{\alpha}\ and\ \mathrm{X}^{\alpha}\cap\mathrm{X}^{\beta}=\varnothing,\ when\ \alpha\neq\beta \tag{4.3}$$

Of course, the cardinality (i.e., the number of members) of each one of these subsets is 36/4 equal to 9.

The above discussion had the sole purpose of motivating the more general and formal developments that follow. So, now we go back to the general case introduced in Section 3, in which the sets of node-indexes and subdomain-indexes are $\hat{\mathrm{N}}=\{1,\ldots,n\}$ and $\hat{E}=\{1,\ldots,E\}$, respectively, and define a *'derived-node'* to be any pair of numbers, $(p,\ \alpha)$, such that $p\in\hat{\mathrm{N}}^{\alpha}$. Then, the total set of *derived-nodes*, fulfills:

$$\mathrm{X}=\{(p,\ \alpha)\,|\,\alpha\in\hat{E}\ \&\ p\in\hat{\mathrm{N}}^{\alpha}\} \tag{4.4}$$

In order to avoid unnecessary repetitions, in the developments that follow where we deal extensively with *derived nodes*, the notation $(p,\ \alpha)$ is reserved for pairs such that $(p,\ \alpha)\in X$. Some subsets of X are defined next:

$$\mathrm{I}\equiv\{(p,\ \alpha)\,|\,p\in\hat{\mathrm{N}}_{\mathrm{I}}\}\ and\ \Gamma\equiv\{(p,\ \alpha)\,|\,p\in\hat{\mathrm{N}}_{\Gamma}\} \tag{4.5}$$

$$\pi\equiv\{(p,\ \alpha)\,|\,p\in\hat{\mathrm{N}}_{\pi}\}\ and\ \Delta\equiv\{(p,\ \alpha)\,|\,p\in\hat{\mathrm{N}}_{\Delta}\} \tag{4.6}$$

With each $\alpha=1,\ldots,E$, we associate a unique *'local subset of derived-nodes'*:

$$\mathrm{X}^{\alpha}\equiv\{(p,\ \alpha)\} \tag{4.7}$$

The family of subsets $\{\mathrm{X}^{1},\ldots,\mathrm{X}^{\mathrm{E}}\}$, is a truly disjoint decomposition of the whole set of *derived-nodes*, in the sense that:

$$\mathrm{X}=\bigcup_{\alpha=1}^{E}\mathrm{X}^{\alpha}\ and\ \mathrm{X}^{\alpha}\cap\mathrm{X}^{\beta}=\varnothing,\ when\ \alpha\neq\beta \tag{4.8}$$

Section 5
The "Derived Vector-Space (DVS)"

Firstly, we recall from Section 3 the definition of the vector space $\widehat{W}$. Then, for each $\alpha=1,\ldots,E$, we define the vector-subspace $\widehat{W}^{\alpha}\subset\widehat{W}$, which is constituted by the vectors that have the property that, for each $i\notin\hat{\mathrm{N}}^{\alpha}$, its *i-component* vanishes. With this notation, the *'product-space' W*, is defined by

$$W\equiv\prod_{\alpha=1}^{E}\widehat{W}^{\alpha}=\widehat{W}^{1}\times\ldots\times\widehat{W}^{E} \tag{5.1}$$

As explained in Section 3, the *'original problem'* of Eq.(3.4) is a problem formulated in the *original vector-space* $\widehat{W}$ and in the developments that follow we transform this problem into one that is formulated in the *product-space* W, which is a space of discontinuous functions.

By a *'derived-vector'* we mean a real-valued function[5] defined in the set X, of *derived-nodes*. The set of *derived-vectors* constitute a linear space, which will be referred to as the *'derived-vector space'*. Corresponding to each *local subset of derived-nodes*, X^{α}, there is a *'local subspace of derived-vectors'*, W^{α}, which is defined by the condition that vectors of W^{α} vanish at every *derived-node* that does not belong to X^{α}. A formal manner of stating this definition is

- $\underline{u}\in W^{\alpha}\subset W$, if and only if, $\underline{u}(p,\beta)=0$ whenever $\beta\neq\alpha$

An important difference between the subspaces W^{α} and $\widehat{W}^{\alpha}$ that should be observed is that $W^{\alpha}\subset W$, while $\widehat{W}^{\alpha}\not\subset W$. In particular,

$$W\equiv\prod_{\alpha=1}^{E}\widehat{W}^{\alpha}=W^{1}\oplus\ldots\oplus W^{E} \tag{5.2}$$

In words: the space W is the product of the family of subspaces $\{\widehat{W}^{1},\ldots,\widehat{W}^{E}\}$, but at the same time it is the *direct-sum* of the family $\{W^{1},\ldots,W^{E}\}$. In view of Eq. (5.2), it is straightforward to establish a *bijection* (actually, an *isomorphism*) between the *derived-vector space* and the *product-space*. Thus, in what follows we identify both.

For every pair of vectors, $\underline{u}\in W$ and $\underline{w}\in W$, the *'Euclidean inner product'* is defined to be

$$\underline{u}\bullet\underline{w}=\sum_{(p,\alpha)\in\mathrm{X}}\underline{u}\left(p,\alpha\right)\underline{w}\left(p,\alpha\right) \tag{5.3}$$

In applications of the theory to systems of equations, when $\underline{u}(p,\ \alpha)$ itself is a vector, Eq. (5.3) is replaced by

$$\underline{u}\bullet\underline{w}=\sum_{(p,\alpha)\in\mathrm{X}}\underline{u}\left(p,\alpha\right)\odot\underline{w}\left(p,\alpha\right) \tag{5.4}$$

Here, $\underline{u}\odot\underline{w}$ means the inner product of the vectors involved. An important property is that the *derived-vector space*, W, constitutes a finite dimensional *Hilbert-space* with respect to the Euclidean inner product. We observe the

[5] For the treatment of systems of equations, such as those of linear elasticity, such functions are vector-valued.

Euclidean inner product independently of the nature of the *original matrix* $\underline{\underline{\hat{A}}}$; in particular it may non-symmetric or indefinite.

The *natural injection*, R: $\widehat{W} \rightarrow W$, of $\widehat{W}$ into W, is defined by the condition that, for every $\underline{\hat{u}} \in \widehat{W}$, one has

$$(R\underline{\hat{u}})(p, \alpha) = \underline{\hat{u}}(p), \forall (p, \alpha) \in \mathrm{X} \qquad (5.5)$$

The *'multiplicity'*, $m(p)$, of any *original-node* $p \in \hat{\mathrm{N}}$ is characterized by the property [Herrera and Yates 2010, Herrera and Yates 2009]:

$$\sum_{\alpha=1}^{E} (R\underline{\hat{u}})(p, \alpha) = m(p)\underline{\hat{u}}(p) \qquad (5.6)$$

The space W will be decomposed into two orthogonal complementary subspaces $W_{11} \subset W$ and $W_{12} \subset W$, so that

$$W = W_{11} + W_{12} \text{ and } \{0\} = W_{11} \cap W_{12} \qquad (5.7)$$

Here, the subspace $W_{12} \subset W$ is the *natural injection* of $\widehat{W}$ into W; i.e.,

$$W_{12} \equiv R\widehat{W} \subset W \qquad (5.8)$$

and $W_{11} \subset W$ its orthogonal complement with respect to the Euclidean inner product. For later use, we point out that the inverse of R: $\widehat{W} \rightarrow W$, when restricted to $W_{12} \subset W$, exists and will denoted by R^{-1}: $W_{12} \rightarrow \widehat{W}$. Here, we recall that it is customary to use the *direct-sum* notation:

$$W = W_{11} \oplus W_{12} \qquad (5.9)$$

when the pair of equalities of Eq. (5.7), holds. The *'subspace of continuous vectors'* is defined to be $W_{12} \subset W$, while the *'subspace of zero-average vectors'* is defined to be $W_{11} \subset W$. Two matrices $\underline{\underline{a}}: W \rightarrow W$ and $\underline{\underline{j}}: W \rightarrow W$ are here introduced; they are the projections operators, with respect to the *Euclidean inner-product*, on W_{12} and W_{11}, respectively. The first one will be referred to as the *'average operator'* and the second one will be the *'jump operator'*, respectively. We observe that in view of Eq. (5.7), every vector, $\underline{u} \in W$, can be uniquely written as the sum of a *zero-average vector* plus a *continuous vector* (we could say: a *zero-jump* vector); indeed:

$$\underline{u} = \underline{u}_{11} + \underline{u}_{12} \; with \begin{cases} \underline{u}_{11} \equiv \underline{\underline{j}}\,\underline{u} \in W_{11} \\ \underline{u}_{12} \equiv \underline{\underline{a}}\,\underline{u} \in W_{12} \end{cases} \qquad (5.10)$$

The vectors $\underline{\underline{j}}\,\underline{u}$ and $\underline{\underline{a}}\,\underline{u}$ are said to be the *'jump'* and the *'average'* of $\underline{u}$, respectively.

The linear subspaces that are defined next are chosen to mimic those used by other authors [Mandel and Dohrmann 2003; Mandel *et al*., 2005]. In particular, W_I, W_Γ, W_π, and W_Δ are defined by imposing the restrictions that follow to their members. Vectors of:

- W_I vanish at every derived-node that is not an internal node;
- W_Γ vanish at every derived-node that is not an interface node;
- W_π vanish at every derived-node that is not a primal node; and
- W_Δ vanish at every derived-node that is not a dual node.

Furthermore,

- $W_r \equiv W_I + \underline{\underline{a}} W_\pi + W_\Delta$;
- $W_\Pi \equiv W_I + \underline{\underline{a}} W_\pi$.

We observe that each one of the following families of subspaces are linearly independent:

$$\{W_I, W_\Gamma\}, \{W_I, W_\pi, W_\Delta\}, \{W_\Pi, W_\Delta\}$$

And also that

$$W = W_I + W_\Gamma = W_I + W_\pi + W_\Delta \; and \; W_r = W_\Pi + W_\Delta \qquad (5.11)$$

The above definition of W_r is appropriate when considering *dual-primal formulations*; other kinds of restrictions require changing the term $\underline{\underline{a}} W_\pi$ by $\underline{\underline{a}}^r W_\pi$, where $\underline{\underline{a}}^r$ is a projection on the restricted subspace.

Section 6
The general problem with constraints

The following result is similar to results shown in [Herrera and Yates 2010; Herrera and Yates 2009]; its proof, as well as the definition of the matrix $\underline{\underline{A}}: W_r \rightarrow W_r$ that is used in it, is given in Appendix "A":

"A vector $\underline{\hat{u}} \in \widehat{W}$ is solution of the *original problem*, if and only if, $\underline{u}' = R\,\underline{\hat{u}} \in W_r \subset W$ fulfills the equalities:

$$\underline{\underline{a}}\,\underline{\underline{A}}\,\underline{u}' = \underline{\overline{f}} \; and \; \underline{\underline{j}}\,\underline{u}' = 0 \qquad (6.1)$$

The vector $\underline{\overline{f}} \equiv \left(R\,\underline{\hat{f}}\right) \in W_{12} \subset W_r$, will be written as $\underline{\overline{f}} \equiv \underline{\overline{f}}_\Pi + \underline{\overline{f}}_\Delta$, with $\underline{\overline{f}}_\Pi \in W_\Pi$ and $\underline{\overline{f}}_\Delta \in W_\Delta$."

This is the *'dual-primal problem formulated in the derived-vector space'*; or, simply, the *DVS-dual-primal problem*. We remark that this

problem is formulated in the subspace W_r of the *derived-vector space* W, in which the restrictions have been incorporated. Thus, all the algorithms to be discussed include such restrictions; in particular, those imposed by means of *primal* nodes. In what follows, the matrix $\underline{\underline{A}}: \widehat{W}_r \rightarrow \widehat{W}_r$ is assumed to be invertible. In many cases this can be granted when a sufficiently large number of primal nodes, adequately located, are taken. Let $\underline{u}' \in W_r$ be solution of it, then $\underline{u}' \in W_{12} \subset W$ necessarily, since $\underline{\underline{j}}\,\underline{u}' = 0$, and one can apply the inverse of the *natural injection* to obtain

$$\underline{\hat{u}} = R^{-1}\,\underline{u}' \tag{6.2}$$

Since this problem is formulated in the *derived-vectors space*, in the algorithms to be presented all the operations are carried out in *such* a space; in particular, we will never return to the *original vector space,* $\widehat{W}$, except at the end when we apply Eq. (6.2).

Section 7
The schur complement algorithm

The matrix $\underline{\underline{A}}$ of Eq. (6.1), can be written as (here, we draw from Appendix "A"):

$$\underline{\underline{A}} = \begin{pmatrix} \underline{\underline{A}}_{\Pi\Pi} & \underline{\underline{A}}_{\Pi\Delta} \\ \underline{\underline{A}}_{\Delta\Pi} & \underline{\underline{A}}_{\Delta\Delta} \end{pmatrix} \tag{7.1}$$

Using this notation, we define the *'dual-primal Schur-complement matrix'* by

$$\underline{\underline{S}} \equiv \underline{\underline{A}}_{\Delta\Delta} - \underline{\underline{A}}_{\Delta\Pi}\,\underline{\underline{A}}_{\Pi\Pi}^{-1}\,\underline{\underline{A}}_{\Pi\Delta} \tag{7.2}$$

Let be $\underline{u} \equiv \underline{u}' - \underline{\underline{A}}_{\Pi\Pi}^{-1}\,\overline{\underline{f}}_{\Pi}$, then Eq. (6.1) is equivalent to: *"Given* $\underline{f} = \underline{f}_{\Delta} \in \underline{\underline{a}} W_{\Delta}$, *find a* $\underline{u}_{\Delta} \in W_{\Delta}$ *such that*

$$\underline{\underline{a}}\,\underline{\underline{S}}\,\underline{u}_{\Delta} = \underline{f}_{\Delta} \;\; and \;\; \underline{\underline{j}}\,\underline{u}_{\Delta} = 0 \tag{7.3}"$$

Here, $\underline{u} = \underline{u}_{\Pi} + \underline{u}_{\Delta}$ and

$$\underline{f}_{\Delta} \equiv \overline{\underline{f}}_{\Delta} - \underline{\underline{A}}_{\Delta\Pi}\,\underline{\underline{A}}_{\Pi\Pi}^{-1}\,\overline{\underline{f}}_{\Pi} \; and \; \underline{u}_{\Pi} \equiv -\underline{\underline{A}}_{\Pi\Pi}^{-1}\,\underline{\underline{A}}_{\Pi\Delta}\,\underline{u}_{\Delta} \tag{7.4}$$

We observe that the *Schur complement matrix*, $\underline{\underline{S}}: W_{\Delta} \rightarrow W_{\Delta}$, is invertible when so is $\underline{\underline{A}}: W_r \rightarrow W_r$ [Herrera and Yates 2010; Herrera and Yates 2009].

In Appendix "B" it is shown that Eq. (7.3) is the discrete version of a *non-preconditioned Dirichlet-Dirichlet problem*. Thus, this algorithm could be called the '*non-preconditioned Dirichlet-Dirichlet algorithm'*. However, in what follows, the algorithm that corresponds to Eq. (7.3) will be referred to as the '*Schur-complement algorithm'*, since it is a variant of one of the simplest forms of substructuring methods described, for example, in [Smith e*t al*., 1996].

Section 8
The dual Neumann-Neumann problem

In this paper we present three alternative procedures for obtaining the algorithm we are about to derive. One is as a Neumann-Neumann formulation, discussed in Appendix "B" using an operator, which is the *counter-part* of the Steklov-Poincaré operator; one more is in what could be called the classical manner that consists in using a Lagrange multipliers treatment of the problem of Section 7 [Toselli and Widlund 2005] (the DVS version of this approach is presented in Appendix "C"); and the third one -used in this Section- stems from the identity

$$\underline{\underline{a}}\,\underline{\underline{S}} + \underline{\underline{j}}\,\underline{\underline{S}} = \underline{\underline{S}} \tag{8.1}$$

This latter equation is clear since $\underline{\underline{a}} + \underline{\underline{j}} = \underline{\underline{I}}$.

Eqs.(7.3) and (8.1), together, imply that

$$\underline{\underline{S}}\,\underline{u}_{\Delta} = \underline{\underline{a}}\,\underline{\underline{S}}\,\underline{u}_{\Delta} + \underline{\underline{j}}\,\underline{\underline{S}}\,\underline{u}_{\Delta} = \underline{f}_{\Delta} - \underline{\lambda} \tag{8.2}$$

when the vector $\underline{\lambda}$ is defined to be

$$\underline{\lambda} \equiv -\underline{\underline{j}}\,\underline{\underline{S}}\,\underline{u}_{\Delta} \tag{8.3}$$

Therefore, $\underline{\lambda} \in \underline{\underline{j}} W_{\Delta}$. Thus, the problem of finding $\underline{u}_{\Delta}$ has been transformed into that of finding the *'Lagrange multiplier'* $\underline{\lambda}$[6], since once $\underline{\lambda}$ is known one can apply $\underline{\underline{S}}^{-1}$ to Eq. (8.2), to obtain

$$\underline{u}_{\Delta} = \underline{\underline{S}}^{-1}\left(\underline{f}_{\Delta} - \underline{\lambda}\right) \tag{8.4}$$

Furthermore, in Eq. (8.4), $\underline{u}_{\Delta} \in \underline{\underline{a}} W_{\Delta}$, so that

$$\underline{\underline{j}}\,\underline{\underline{S}}^{-1}\left(\underline{f}_{\Delta} - \underline{\lambda}\right) = 0 \tag{8.5}$$

Hence, $\underline{\lambda} \in W_{\Delta}$ fulfills

$$\underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{\lambda} = \underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{f}_{\Delta} \; together \; with \; \underline{\underline{a}}\,\underline{\lambda} = 0 \tag{8.6}$$

Thereby, we mention that $\underline{\underline{j}}\,\underline{\underline{S}}\,\underline{u}_{\Delta}$ is discretized version of the of the *average* of the normal derivative [Herrera and Yates 2010; Herrera and Yates 2009].

[6] In the Appendix C, it is shown that $\underline{\lambda}$ is indeed the Lagrange multiplier when such an approach is adopted.

Section 9
The primal Neumann-Neumann problem

In Appendix "B", it was shown that there is a second and more direct manner of formulating the non-preconditioned Neumann-Neumann problem, which is given by Eq.(18.28). Here, we derive it for the general problem we are considering. Our starting point will be Eq.(7.3).

We multiply the first equality in Eq. (7.3) by $\underline{\underline{S}}^{-1}$, observing that $\underline{\underline{a}}\underline{f}_{\Delta} = \underline{f}_{\Delta}$, to obtain

$$\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\underline{S}}\underline{u}_{\Delta} = \underline{\underline{S}}^{-1}\underline{f}_{\Delta} = \underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{f}_{\Delta} = \underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\underline{S}}\left(\underline{\underline{S}}^{-1}\underline{f}_{\Delta}\right) \tag{9.1}$$

Thus, Eq. (7.3) can be transformed into

$$\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\underline{S}}\left(\underline{u}_{\Delta} - \underline{\underline{S}}^{-1}\underline{f}_{\Delta}\right) = 0 \;\; and \;\; \underline{\underline{j}}\underline{u}_{\Delta} = 0 \tag{9.2}$$

or

$$\underline{\underline{a}}\underline{\underline{S}}\left(\underline{u}_{\Delta} - \underline{\underline{S}}^{-1}\underline{f}_{\Delta}\right) = 0 \;\; and \;\; \underline{\underline{j}}\underline{u}_{\Delta} = 0 \tag{9.3}$$

If we define:

$$\underline{v}_{\Delta} \equiv \underline{\underline{S}}^{-1}\underline{f}_{\Delta} - \underline{u}_{\Delta} \tag{9.4}$$

Eq. (9.3) is transformed into:

$$\underline{\underline{j}}\underline{v}_{\Delta} = \underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \;\; and \;\; \underline{\underline{a}}\underline{\underline{S}}\underline{v}_{\Delta} = 0 \tag{9.5}$$

The iterative form of this algorithm is obtained multiplying by $\underline{\underline{S}}^{-1}$:

$$\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{v}_{\Delta} = \underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \;\; and \;\; \underline{\underline{a}}\underline{\underline{S}}\underline{v}_{\Delta} = 0 \tag{9.6}$$

If the solution of Eq. (7.3) is known, then $\underline{v}_{\Delta} \in W_{\Delta}$ defined by Eq. (9.4) fulfills Eq. (9.6); conversely, if $\underline{v}_{\Delta} \in W_{\Delta}$ satisfies Eq. (9.6), then

$$\underline{u}_{\Delta} \equiv \underline{\underline{S}}^{-1}\underline{f}_{\Delta} - \underline{v}_{\Delta} \tag{9.7}$$

is solution of Eq. (7.3). We shall refer to the iterative algorithm defined by Eq. (9.6) as the *'multipliers-free formulation of the non-preconditioned Neumann-Neuman problem'*.

Section 10
The second dual Neumann-Neumann problem

Our starting point will be Eq. (8.6). Firstly, we observe the following identity:

$$\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\left(\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\right) = \underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1} \tag{10.1}$$

Then, we multiply the first equality in Eq. (8.6) by $\underline{\underline{S}}$ to obtain

$$\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\left(\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\right)\underline{\lambda} = \underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\lambda} = \underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\left(\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\right)\underline{f}_{\Delta}$$

$$together\ with\ \underline{\underline{a}}\underline{\lambda} = 0 \tag{10.2}$$

Or

$$\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\left(\left(\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\right)\underline{f}_{\Delta} - \underline{\lambda}\right) = 0$$

$$together\ with\ \underline{\underline{a}}\underline{\lambda} = 0 \tag{10.3}$$

If we multiply the first of these equalities by $\underline{\underline{S}}^{-1}$ and define:

$$\underline{\mu} \equiv \underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} - \underline{\lambda} \tag{10.4}$$

Eq. (10.3) is transformed into:

$$\underline{\underline{a}}\underline{\mu} = \underline{\underline{a}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \;\; and \;\; \underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\mu} = 0 \tag{10.5}$$

We observe that this latter equation is equivalent to Eq. (10.3) because $\underline{\underline{S}}^{-1}$ is non-singular. If the solution of Eq. (8.6) is known, then $\underline{\mu} \in W_{\Delta}$ defined by Eq. (10.4) fulfills Eq. (10.5). Conversely, if $\underline{\mu} \in W_{\Delta}$ satisfies Eq. (10.5), then

$$\underline{\lambda} \equiv \underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} - \underline{\mu} \tag{10.6}$$

is solution of Eq. (8.6).

We notice that Eq. (10.5) does not define an iterative algorithm. However, multiplying Eq. (10.5) by $\underline{\underline{S}}$ an iterative algorithm is obtained:

$$\underline{\underline{S}}\underline{\underline{a}}\underline{\mu} = \underline{\underline{S}}\underline{\underline{a}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \;\; and \;\; \underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\mu} = 0 \tag{10.7}$$

Eq. (10.7) supplies an alternative manner of applying the Lagrange-multipliers approach. The equality $\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\mu} = 0$ may be interpreted as a restriction; indeed, It can be shown that it is equivalent to $\underline{\mu} \in \underline{\underline{S}}\underline{\underline{a}}W_{\Delta}$.

Section 11
The DVS version of the BDDC algorithm

The *DVS* version of the *BDDC* is obtained when the *Schur-complement* algorithm, of Section 7, is preconditioned by means of the matrix $\underline{\underline{a}}\underline{\underline{S}}^{-1}$. It is: "Given $\underline{f}_{\Delta} \in \underline{\underline{a}}W_{\Delta}$, find $\underline{u}_{\Delta} \in W_{\Delta}$ such that

$$\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\underline{S}}\underline{u}_{\Delta} = \underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \;\; and \;\; \underline{\underline{j}}\underline{u}_{\Delta} = 0 \tag{11.1}"$$

The following properties should be noticed:

a) This is an iterative algorithm;

b) The iterated matrix is $\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\underline{S}}$;

c) The iteration is carried out in the subspace $\underline{\underline{a}}W_{\Delta} \subset W_{\Delta}$;

d) This algorithm is applicable whenever the Schur complement matrix $\underline{\underline{S}}$ is such that the logical implication is fulfilled, for any $\underline{w} \in W_{\Delta}$:

$$\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{w} = 0 \; and \; \underline{\underline{j}}\underline{w} = 0 \Rightarrow \underline{w} = 0 \quad (11.2)$$

e) In particular, it is applicable when $\underline{\underline{S}}$ is definite.

Properties $a)$ to $c)$ are interrelated. The condition $\underline{\underline{j}}\underline{u}_{\Delta} = 0$ is equivalent to $\underline{u}_{\Delta} \in \underline{\underline{a}} W_{\Delta}$; thus the search is carried out in $\underline{\underline{a}} W_{\Delta}$. When the matrix $\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\underline{S}}$ is applied repeatedly, one remains in $\underline{\underline{a}} W_{\Delta}$, because for every $\underline{w} \in W_{\Delta}$, one has $\underline{\underline{j}}\left(\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\underline{S}}\underline{w}\right) = 0$. As for property $d)$, it means that when the implication of Eq. (11.2) holds, Eq. (11.1) implies Eq. (7.3). To prove this, observe that

$$\underline{\underline{j}}\left(\underline{\underline{a}}\underline{\underline{S}}\underline{u}_{\Delta} - \underline{f}_{\Delta}\right) = 0 \quad (11.3)$$

and also that Eq. (11.1) implies

$$\underline{\underline{a}}\underline{\underline{S}}^{-1}\left(\underline{\underline{a}}\underline{\underline{S}}\underline{u}_{\Delta} - \underline{f}_{\Delta}\right) = 0 \quad (11.4)$$

When the implication of Eq. (11.2) holds, Eqs. (11.3) and (11.4) together imply

$$\underline{\underline{a}}\underline{\underline{S}}\underline{u}_{\Delta} - \underline{f}_{\Delta} = 0 \quad (11.5)$$

As desired, this proves that Eqs. (11.1) implies Eq. (7.3), when Eq. (11.2) holds.

The condition of Eq. (11.2), is weaker than that of (or generalizes that of) requiring that the Schur complement matrix $\underline{\underline{S}}$ be definite, since the implication of Eq. (11.2) is always satisfied when $\underline{\underline{S}}$ is definite. Assume that $\underline{\underline{S}}$ is definite, then for any vector $\underline{w} \in W_{\Delta}$ such that $\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{w} = 0$ and $\underline{\underline{j}}\underline{w} = 0$, one has

$$\underline{w} \bullet \underline{\underline{S}}^{-1}\underline{w} = \underline{w} \bullet \underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{w} = \left(\underline{\underline{j}}\underline{w}\right) \bullet \underline{\underline{S}}^{-1}\underline{w} = 0 \quad (11.6)$$

This implies $\underline{w} = 0$, because $\underline{\underline{S}}^{-1}$ is definite when so is $\underline{\underline{S}}$. Thereby, Property $e)$ is clear.

Section 12
The DVS version of FETI-DP algorithm

The *DVS* version of the *FETI-DP algorithm* is obtained when the *'Lagrange-Multipliers formulation of the non-preconditioned Neumann-Neuman problem'*, of Section 8, Eq. (8.6), is preconditioned by means of the matrix $\underline{\underline{j}}\underline{\underline{S}}$. It is: "Given $\underline{f}_{\Delta} \in \underline{\underline{a}} W_{\Delta}$, find $\underline{\lambda} \in W_{\Delta}$ such that

$$\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\lambda} = \underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \; and \; \underline{\underline{a}}\underline{\lambda} = 0 \quad (12.1)''$$

For this algorithm the following properties should be noticed:

i. This is an iterative algorithm;
ii. The iterated matrix is $\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}$;
iii. The iteration is carried out in the subspace $\underline{\underline{j}} W_{\Delta} \subset W_{\Delta}$;
iv. The algorithm is applicable whenever the Schur complement matrix $\underline{\underline{S}}$ is such that the logical implication is fulfilled, for any $\underline{w} \in W_{\Delta}$:

$$\underline{\underline{j}}\underline{\underline{S}}\underline{w} = 0 \; and \; \underline{\underline{a}}\underline{w} = 0 \Rightarrow \underline{w} = 0 \quad (12.2)$$

v. In particular, it is applicable when $\underline{\underline{S}}$ is positive definite.

Properties *i*) to *ii*) are interrelated. The condition $\underline{\underline{a}}\underline{\lambda} = 0$ is equivalent to $\underline{\lambda} \in \underline{\underline{j}} W_{\Delta}$; thus the search is carried out in $\underline{\underline{j}} W_{\Delta}$. When the matrix $\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}$ is applied repeatedly one remains in $\underline{\underline{j}} W_{\Delta}$, because for every $\underline{\mu} \in W_{\Delta}$, one has $\underline{\underline{a}}(\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\mu}) = 0$. As for property *iv*), it means that when the implication of Eq. (12.2) holds, Eq. (12.1) implies Eq. (8.6). To prove this, assume Eq. (12.1) and observe that

$$\underline{\underline{a}}\left(\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\lambda} - \underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta}\right) = 0 \quad (12.3)$$

and also that Eq. (12.1) implies

$$\underline{\underline{j}}\underline{\underline{S}}\left(\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\lambda} - \underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta}\right) = 0 \quad (12.4)$$

When the implication of Eq. (12.2) holds, Eqs. (12.3) and (12.4) together imply

$$\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\lambda} - \underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} = 0 \quad (12.5)$$

As desired, this proves that Eqs. (12.1) implies Eq. (8.6), when Eq. (12.2) holds.

The condition of Eq. (12.2), is weaker than that of (or generalizes that of) requiring that the Schur complement matrix $\underline{\underline{S}}$ be definite, since the implication of Eq. (12.2) is always satisfied when $\underline{\underline{S}}$ is definite. Indeed, assume that $\underline{\underline{S}}$ is definite, then for any vector $\underline{\mu} \in W_{\Delta}$ such that $\underline{\underline{j}}\underline{\underline{S}}\underline{\mu} = 0$ and $\underline{\underline{a}}\underline{\mu} = 0$, one has

$$\underline{\mu} \bullet \underline{\underline{S}}\underline{\mu} = \underline{\mu} \bullet \underline{\underline{a}}\underline{\underline{S}}\underline{\mu} = \left(\underline{\underline{a}}\underline{\mu}\right) \bullet \underline{\underline{S}}\underline{\mu} = 0 \quad (12.6)$$

This implies $\underline{\mu} = 0$, because $\underline{\underline{S}}$ is definite. Thereby, Property *v*) is clear.

Section 13
Preconditioned primal Neumann-Neumann algorithm

This algorithm is a preconditioned version of the *multipliers-free formulation of the non-preconditioned Neumann-Neumann problem*. It can be derived multiplying the first equality in Eq. (9.5) by the preconditioner $\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}$. Thus, such an algorithm consists in searching for a function $\underline{v}_{\Delta}\in W_{\Delta}$, which fulfills

$$\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{v}_{\Delta}=\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta}\ \ and\ \underline{\underline{a}}\underline{\underline{S}}\underline{v}_{\Delta}=0 \tag{13.1}$$

For this algorithm the following properties should be noticed:

A. This is an iterative algorithm;
B. The iterated matrix is $\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}$;
C. The iteration is carried out in the subspace $\underline{\underline{S}}^{-1}\underline{\underline{j}}W_{\Delta}\subset W_{\Delta}$;
D. The algorithm is applicable whenever the Schur complement matrix $\underline{\underline{S}}$ is such that the logical implication is fulfilled, for any $\underline{w}\in W_{\Delta}$:

$$\underline{\underline{j}}\underline{\underline{S}}\underline{w}=0\ and\ \underline{\underline{a}}\underline{w}=0\Rightarrow\underline{w}=0 \tag{13.2}$$

E. In particular, it is applicable when $\underline{\underline{S}}$ is positive definite.

Properties A) to C) are interrelated. The condition $\underline{\underline{a}}\underline{\underline{S}}\underline{v}_{\Delta}=0$ is equivalent to $\underline{v}_{\Delta}\in\underline{\underline{j}}\underline{\underline{S}}^{-1}W_{\Delta}$; thus the search is carried out in the subspace $\underline{\underline{j}}\underline{\underline{S}}^{-1}W_{\Delta}$. When the matrix $\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}$ is applied repeatedly one remains in $\underline{\underline{j}}\underline{\underline{S}}^{-1}W_{\Delta}$, because for every $\underline{v}_{\Delta}\in W_{\Delta}$, one has $\underline{\underline{a}}\underline{\underline{S}}(\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{v}_{\Delta})=0$. As for property D), it means that when the implication of Eq. (13.2) holds, Eq. (13.1) implies Eq. (9.5). To prove this, assume Eq. (13.1) and define

$$\underline{w}\equiv\underline{\underline{j}}\underline{v}_{\Delta}-\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta}\ so\ that\ \underline{\underline{a}}\underline{w}=0 \tag{13.3}$$

Furthermore, in view of Eq. (13.1)

$$\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{w}=0\ and\ therefore\ \underline{\underline{j}}\underline{\underline{S}}\underline{w}=0 \tag{13.4}$$

Using Eq. (13.2), it is seen that Eqs. (13.3) and (13.4) together imply

$$\underline{\underline{j}}\underline{v}_{\Delta}-\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta}=\underline{w}=0 \tag{13.5}$$

Now Eq. (9.5) is clear and the proof is complete.

The condition of Eq. (13.2), is weaker than (or generalizes that of) requiring that the Schur complement matrix $\underline{\underline{S}}$ be definite, since the implication of Eq. (13.2) is always satisfied when $\underline{\underline{S}}$ is definite. Indeed, assume that $\underline{\underline{S}}$ is definite, then for any vector $\underline{w}\in W_{\Delta}$ such that $\underline{\underline{j}}\underline{\underline{S}}\underline{w}=0$ and $\underline{\underline{a}}\underline{w}=0$, one has

$$\underline{w}\bullet\underline{\underline{S}}\underline{w}=\underline{w}\bullet\underline{\underline{a}}\underline{\underline{S}}\underline{w}=\left(\underline{\underline{a}}\underline{w}\right)\bullet\underline{\underline{S}}\underline{w}=0 \tag{13.6}$$

This implies $\underline{w}=0$, because $\underline{\underline{S}}$ is definite. Thereby, Property *E*) is clear.

Section 14
Preconditioned second dual Neumann-Neumann algorithm

This algorithm is a preconditioned version of the *'second form of the Lagrange-Multipliers formulation of the non-preconditioned Neumann-Neuman problem'*, of Section 10. Multiplying Eq. (10.5) by the matrix $\underline{\underline{S}}\underline{\underline{a}}\underline{\underline{S}}^{-1}$, we obtain:

$$\underline{\underline{S}}\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\mu}=\underline{\underline{S}}\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta}\ \ and\ \underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\mu}=0 \tag{14.1}$$

The iterative non-overlapping algorithm that is obtained by the use of Eq. (14.1) is similar to FETI-DP. The following properties should be noticed:

I. Firstly, this is an iterative algorithm;
II. The iterated matrix is $\underline{\underline{S}}\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}$;
III. The iteration is carried out in the subspace $\underline{\underline{S}}\underline{\underline{a}}W_{\Delta}\subset W_{\Delta}$;
IV. This algorithm is applicable whenever the Schur complement matrix $\underline{\underline{S}}$ is such that the logical implication is fulfilled, for any $\underline{w}\in W_{\Delta}$:

$$\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{w}=0\ and\ \underline{\underline{j}}\underline{w}=0\Rightarrow\underline{w}=0 \tag{14.2}$$

V. In particular, it is applicable when $\underline{\underline{S}}$ is positive definite in W_{Δ}.

Properties *I*) to *III*) are interrelated. The condition $\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\mu}=0$ is equivalent to $\underline{\mu}\in\underline{\underline{S}}\underline{\underline{a}}W_{\Delta}$; thus the search is carried out in the subspace $\underline{\underline{S}}\underline{\underline{a}}W_{\Delta}$. When the matrix $\underline{\underline{S}}\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}$ is applied repeatedly one remains in $\underline{\underline{S}}\underline{\underline{a}}W_{\Delta}$, because for every $\underline{\mu}\in W_{\Delta}$, one has $\underline{\underline{a}}\underline{\underline{S}}(\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\mu})=0$. As for property *IV*), it means that when the implication of Eq. (14.2) holds, Eq. (14.1) implies Eq. (10.5). To prove this, assume Eq. (14.1) and define

$$\underline{\eta} \equiv \underline{\underline{a}}\,\underline{\mu} - \underline{\underline{a}}\,\underline{\underline{S}}\,\underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{f}_{\Delta} \; so\ that\ \underline{\underline{j}}\,\underline{\eta} = 0 \quad (14.3)$$

Furthermore, in view of Eq. (14.1)

$$\underline{\underline{S}}\,\underline{\underline{a}}\,\underline{\underline{S}}^{-1}\,\underline{\eta} = 0 \quad (14.4)$$

Eqs. (14.3) and (14.4), together, imply

$$\underline{\underline{a}}\,\underline{\mu} - \underline{\underline{a}}\,\underline{\underline{S}}\,\underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{f}_{\Delta} = \underline{\eta} = 0 \quad (14.5)$$

Now Eq. (10.5) is clear and, as desired, it has been shown that Eq. (14.1) implies Eq. (10.5), when the condition of Eq. (14.2) holds.

The condition of Eq. (14.2), is weaker than (or generalizes that of) requiring that the Schur complement matrix $\underline{\underline{S}}$ be definite, in the sense that any positive definite matrix fulfills it. Indeed, assume that $\underline{\underline{S}}$ is definite, then for any vector $\underline{w} \in W_{\Delta}$ such that $\underline{\underline{a}}\,\underline{\underline{S}}^{-1}\,\underline{w} = 0$ and $\underline{\underline{j}}\,\underline{w} = 0$, one has

$$\underline{w} \bullet \underline{\underline{S}}^{-1}\,\underline{w} = \underline{w} \bullet \underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{w} = \left(\underline{\underline{j}}\,\underline{w}\right) \bullet \underline{\underline{S}}^{-1}\,\underline{w} = 0 \quad (14.6)$$

This implies $\underline{w} = 0$, because $\underline{\underline{S}}^{-1}$ is definite when so is $\underline{\underline{S}}$. Thereby, Property *V*) is clear.

Section 15
FETI-DP and BDDC from the DVS perspective

As said in the Introduction, both the FETI-DP and BDDC can be accommodated in the DVS-framework. In this Section, we show that the DVS version of FETI-DP presented in Section 12, is obtained when suitable choices are made in the general expressions of FETI-DP. As for BDDC, its relation with the algorithm presented in Section 9 is a little more complicated.

FETI-DP

The FETI preconditioner is given by

$$\widehat{M}_r^{\ -1} = B_{D_r} S B_{D_r}^T = \sum_{i=1}^{N} D_r^{(i)} B_r^{(i)} S^{(i)} B_r^{(i)T} D_r^{(i)} \quad (15.1)$$

and we now have to solve the preconditioned system

$$P_r \widehat{M}_r^{\ -1} P_r^T F_r \lambda_r = P_r \widehat{M}_r^{\ -1} P_r^T d_r \quad (15.2)$$

were $F_r = B_r S^{\dagger} B_r^T$ and $d_r = B_r S^{\dagger} f$ (see, page 157, Eq. (6.51) and (6.52) of [Toselli and Widlund 2005]). Developing the expression and replacing $\widehat{M}_r^{\ -1}$ we get

$$P_r \left(B_{D_r} S B_{D_r}^T\right) P_r^T \left(B_r S^{\dagger} B_r^T\right) \lambda_r = P_r \left(B_{D_r} S B_{D_r}^T\right) P_r^T \left(B_r S^{\dagger} f_r\right). \quad (15.3)$$

We take $P_r = \underline{\underline{I}}$. So, we have

$$B_{D_r} S B_{D_r}^T B_r S^{\dagger} B_r^T \lambda_r = B_{D_r} S B_{D_r}^T B_r S^{\dagger} f_r \quad (15.4)$$

replacing $B_{D_r} = D_r B_r$, get

$$D_r B_r S \left(D_r B_r\right)^T B_r S^{\dagger} B_r^T \lambda_r = D_r B_r S \left(D_r B_r\right)^T B_r S^{\dagger} f_r \quad (15.5)$$

simplifying

$$D_r B_r S B_r^T D_r B_r S^{\dagger} B_r^T \lambda_r = D_r B_r S B_r^T D_r B_r S^{\dagger} f_r. \quad (15.6)$$

Now, we choose $B_r \equiv \underline{\underline{j}}$ and $D_r \equiv \underline{\underline{I}}$, so that $B_r^T = \underline{\underline{j}}$, to obtain

$$\underline{\underline{j}}\,\underline{\underline{S}}\,\underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{\lambda}_r = \underline{\underline{j}}\,\underline{\underline{S}}\,\underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{f}_r \quad (15.7)$$

One advantage of introducing $\underline{\underline{j}}$ is its very convenient algebraic properties; for example, it is idempotent. In particular, here we have used the fact that $\underline{\underline{j}}\,\underline{\lambda}_r = \underline{\lambda}_r$, since $\underline{\underline{a}}\,\underline{\lambda}_r = 0$. Except for slight changes of notation this is the same as Eq. (12.1).

BDDC

In the standard notation used in BDDC [Dohrmann 2003; Mandel and Dohrmann 2003; Mandel *et al*., 2005; Da Conceição 2006]

$$M^{-1} S u = M^{-1} f \quad (15.8)$$

where S and the preconditioner M^{-1} are

$$S = \sum_{i=1}^{N} \overline{R}_i^{\ T} S_i \overline{R}_i \ and\ M^{-1} = \sum_{i=1}^{N} R^T{}_i S_i^{-1} R_i \quad (15.9)$$

respectively. Furthermore, N is the number of subdomains and for each $i = 1, \ldots, N$

$$S_i = A^i{}_{\Gamma\Gamma} - A^i{}_{\Gamma I} \left(A^i{}_{II}\right)^{-1} A^i{}_{I\Gamma}, \ for\ each\ i = 1, \ldots, N \quad (15.10)$$

$R_i = \Gamma \to \Gamma_i$ is the restriction operator from Γ into Γ_i; when applied to a function defined in Γ, it yields its restriction to Γ_i. As for $\overline{R}_i$, $\overline{R}_i : \Gamma \to \Gamma_i$ is given by $\overline{R}_i \equiv D_i R_i$. Here, $D_i = diag\{\delta_i\}$ is a diagonal matrix defining a partition of unity. Substituting S and M^{-1} in Eq. (15.8), we obtain

$$\left(\sum_{i=1}^{N} \left(D_i^{-1} \overline{R}_i\right)^T S_i^{-1} D_i^{-1} \overline{R}_i\right) \left(\sum_{i=1}^{N} \overline{R}_i^{\ T} S_i \overline{R}_i\right) u = \left(\sum_{i=1}^{N} \left(D_i^{-1} \overline{R}_i\right)^T S_i^{-1} D_i^{-1} \overline{R}_i\right) f \quad (15.11)$$

This equation is to be compared with our Eq. (9.1). For the purpose of comparison, the vectors u and f of Eq. (15.11) can be identified with vectors $\underline{u}$ and $\underline{f}$ of our *original space*, $\widehat{W}$. Furthermore, we apply our *natural injection*, $R : \widehat{W} \rightarrow W$, defined by Eq.(5.5), to Eq.(9.1) and pre-multiply the resulting equation also by the *natural injection*, with $\underline{u}$ and $\underline{f}_{\Delta}$ replaced by $R\underline{u}$ and $R\underline{f}$, respectively. In this manner we obtain.

$$R\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\underline{S}}R\underline{u} = \underline{\underline{a}}\underline{\underline{S}}^{-1}R\underline{f} \qquad (15.12)$$

We have verified that indeed Eqs. (15.11) and (15.12) are equivalent.

Comparisons

The DVS approach, and therefore also the DVS-versions of FETI-DP and BDDC here presented, starts with the matrix that is obtained after the problem has been discretized and for its application does not require any information about the system of partial differential equations from which it originated. Generally, all the non-preconditioned DVS-algorithms that have been presented throughout this paper are equally applicable to symmetric, indefinite and non-symmetric matrices. The specific conditions required for its applicability are spelled out in detail in [Herrera and Yates 2009] (Section 9). Throughout all the developments it is assumed that the *dual-primal Schur-complement matrix* $\underline{\underline{S}}$, defined in Section 7, Eq.(7.2), is non-singular.

As said before, for FETI we show that the DVS-version of FETI-DP can be obtained when suitable choices are made in the general expressions of FETI-DP. Although, these choices represent particular case of the general FETI-DP algorithm, in some sense the choices made are optimal because both $\underline{\underline{a}}$ and $\underline{\underline{j}}$ are complementary orthogonal projections, as it has been verified numerically in [Herrera and Yates 2010; Herrera and Yates 2009] and other more recent numerical implementations. When carrying out the incorporation of BDDC in the *DVS-framework* we encountered more substantial differences. For example, when the inverses of the local Schur-complements exist, in the *DVS framework* the inverse of $\underline{\underline{S}}^t$ is given by (see Appendix "A"):

$$\left(\underline{\underline{S}}^t\right)^{-1} = \sum_{\alpha=1}^{N}\left(\underline{\underline{S}}^{\alpha}\right)^{-1} \qquad (15.13)$$

A similar relation does not hold for the BDDC algorithm. Indeed, in this latter approach in that case we have instead:

$$S = \sum_{\alpha=1}^{N}\overline{R}_{\alpha}^{T}S_{\alpha}\overline{R}_{\alpha} \qquad (15.14)$$

and

$$\left(S\right)^{-1} \neq \sum_{\alpha=1}^{N}\left(\overline{R}_{\alpha}^{T}S_{\alpha}\overline{R}_{\alpha}\right)^{-1} \qquad (15.15)$$

even when the inverses of the local Schur-complements exist and no restrictions are used.

The origin of this problem, encountered in the BDDC formulation, may be traced back to the fact that the BDDC approach does not work directly in the product space. Indeed, one frequently goes back to degrees of freedom associated with the *original nodes*. This is done by means of the restriction operators $R_i : \Gamma \rightarrow \Gamma_i$ which can be interpreted as transformations of the *original vector-space* into the *product vector-space* (or, *derived vector-space*). If the algorithm of Eq. (15.11) is analyzed from this point of view, it is seen that it repeatedly goes from the *original vector-space* to the product-space (or *derived vector-space*) and back. For example, consider the expression:

$$\left(\sum_{i=1}^{N}\left(D_i^{-1}\overline{R}_i\right)^{T}S_i^{-1}D_i^{-1}\overline{R}_i\right)\left(\sum_{i=1}^{N}\overline{R}_i^{T}S_i\overline{R}_i\right)u = \left(\sum_{i=1}^{N}\left(D_i^{-1}\overline{R}_i\right)^{T}S_i^{-1}D_i^{-1}\overline{R}_i\right)\left(\sum_{i=1}^{N}\overline{R}_i^{T}S_i\overline{R}_i u\right) \qquad (15.16)$$

occurring in Eq. (15.11). After starting with the vector u in the *original vector-space*, we go to the *derived-vector space* with $\overline{R}_i u$ and remain there when we apply S_i. However, we go back to the *original vector-space* when $\overline{R}_i^T$ is applied. A similar analysis can be made of the term

$$\left(D_i^{-1}\overline{R}_i\right)^{T}S_i^{-1}D_i^{-1}\overline{R}_i \qquad (15.17)$$

Summarizing, in the operations indicated in Eq. (15.16) four trips between the *original vector space* and the *derived-vector space* were made, two one way and the other two in the way back. In the *DVS-framework*, on the other hand, from the start the *original problem* is transformed into one defined in *derived-vector space*, where all the work is done afterwards, and that permits avoiding all those unnecessary trips. Thereby, the matrix formulas are simplified and so is code development. The unification and simplification achieved in this manner, permits producing more effective and robust software.

Section 16
Conclusions and discusions

1. A *primal* framework for the formulation of non-overlapping domain decomposition methods

has been proposed, which is referred to as *'the derived-vector space (DVS) framework'*;

2. *Dual* and *primal* formulations have been derived in a unified manner. Symmetric, non-symmetric and indefinite matrices are also included. Furthermore, detailed conditions that such matrices need to satisfy in order for the general algorithms to be applicable to them have been given in Section 9 of [Herrera and Yates 2009] and in Sections 7 to 14 of the present paper;

3. A brief and effective summary of non-overlapping domain decomposition methods has been obtained. It consists of eight matrix-formulas: four are *primal* formulations and the other four are *dual* formulations;

4. The non-preconditioned formulas are:

$$Dirichlet\text{-}Dirichlet\begin{cases}\underline{\underline{a}}\underline{\underline{S}}\underline{u}_{\Delta}=\underline{f}_{\Delta} \;\; and \;\; \underline{\underline{j}}\underline{u}_{\Delta}=0, \; primal\end{cases} \tag{16.1}$$

$$Neumann\text{-}Neumann$$

$$\begin{cases}\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{v}_{\Delta}=\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \;\; and \;\; \underline{\underline{a}}\underline{\underline{S}}\underline{v}_{\Delta}=0, \; primal\\ \underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\lambda}=\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \; together\; with \; \underline{\underline{a}}\underline{\lambda}=0, dual\;\#1\\ \underline{\underline{S}}\underline{\underline{a}}\underline{\mu}_{\Delta}=\underline{\underline{S}}\underline{\underline{a}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \;\; and \;\; \underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\mu}_{\Delta}=0, dual\;\#2\end{cases} \tag{16.2}$$

5. The preconditioned formulas are:

$$DVS\text{-}BDDC\begin{cases}\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\underline{S}}\underline{u}_{\Delta}=\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \;\; and \;\; \underline{\underline{j}}\underline{u}_{\Delta}=0, \; primal\#1\end{cases} \tag{16.3}$$

$$DVS-FETI-DP\begin{cases}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\lambda}_{\Delta}=\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \;\; and \;\; \underline{\underline{a}}\underline{\lambda}_{\Delta}=0, dual\#1\end{cases} \tag{16.4}$$

$$Neumann\text{-}Neumann$$

$$\begin{cases}\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{v}_{\Delta}=\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \;\; and \;\; \underline{\underline{a}}\underline{\underline{S}}\underline{v}_{\Delta}=0, \; primal\#2\\ \underline{\underline{S}}\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\mu}_{\Delta}=\underline{\underline{S}}\underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta} \;\; and \;\; \underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\mu}_{\Delta}=0, dual\;\#2\end{cases} \tag{16.5}$$

6. The most commonly used methods, BDDC and FETI-DP, have been incorporated in this framework, producing in this manner *DVS-versions* of such methods. Eq. (16.1), is the *primal* formulation of a *Dirichlet-Dirichlet* problem; when this is preconditioned the *DVS-BDDC* is obtained. The formulation of a *Neumann-Neumann* problem using the counter-part of the Steklov-Poincaré operator, given in Appendix "B", yields a *dual formulation*, which is stated in the second equality of Eq. (16.2). The *DVS version* of FETI-DP, of Eq. (16.4), is the preconditioned form of this formulation;

7. For the other matrix-formulas, two preconditioned and two more non-preconditioned, we have not been able to find suitable counterparts in the DDM literature already published;

8. Using the detailed definitions given in [Herrera and Yates 2010; Herrera and Yates 2009], the above *DVS* formulas can be used directly for code development. They are somewhat simpler than those of the BDDC framework and have permitted us to simplify code-development and also to develop very robust computational codes for the examples considered in [Herrera and Yates 2010; Herrera and Yates 2009].

FETI- DP and BDDC are optimal in the sense that the condition number κ of its interface problem grows asymptotically as [Dohrmann 2003; Klawonn and Widlund 2001; Tezaur 1998]:

$$\kappa = O\left(1+\log^2\left(H/h\right)\right) \tag{16.6}$$

Furthermore, they perform quite similarly when the same set of primal constraints is used. Therefore, to be competitive the last two preconditioned *Dirichlet-Dirichlet* algorithms of Eq. (16.5) should have a similar behavior, but at present that is an open question.

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Appendix "A"
Construction of the matrix $\underline{\underline{A}}: W_r \to W_r$

Let the *original matrix* $\underline{\underline{\widehat{A}}}: \widehat{W} \to \widehat{W}$ be written as

$$\underline{\underline{\widehat{A}}} = \left(\widehat{A}_{pq} \right) \tag{17.1}$$

For every pair (p,q) such that $p \in \hat{\mathrm{N}}$ and $q \in \hat{\mathrm{N}}$ we define

$$\delta_{pq}^{\alpha} \equiv \begin{cases} 1, if\ p,q \in \hat{\mathrm{N}}^{\alpha} \\ 0, if\ p \notin \hat{\mathrm{N}}^{\alpha}\ or\ q \notin \hat{\mathrm{N}}^{\alpha} \end{cases}, \alpha = 1,...,E \tag{17.2}$$

Together with

$$m(p,q) \equiv \sum_{\alpha=1}^{E} \delta_{pq}^{\alpha}\ and\ s(p,q) \equiv \begin{cases} 1, when\ m(p,q) = 0 \\ m(p,q), when\ m(p,q) \neq 0 \end{cases} \tag{17.3}$$

The function $m(p,q)$ is said to be the '*multiplicity*' of the pair (p, q). It can be seen that due to the *basic assumption* of Eq. (3.9), we have

$$m(p,q) = 0 \ \Rightarrow \widehat{A}_{pq} = 0 \tag{17.4}$$

We define

$$\underline{\underline{\widehat{A}}}^{\alpha} \equiv \left(\widehat{A}_{pq}^{\alpha} \right) with\ \widehat{A}_{pq}^{\alpha} \equiv \frac{\widehat{A}_{pq} \delta_{pq}^{\alpha}}{s(p,q)} \tag{17.5}$$

With these definitions, we observe the identity

$$\underline{\underline{\widehat{A}}} = \sum_{\alpha=1}^{E} \underline{\underline{\widehat{A}}}^{\alpha} \tag{17.6}$$

For each $\gamma = 1,...,E$, define the matrix $\underline{\underline{A}}^{\gamma}: W \to W$, by:

$$\underline{\underline{A}}^{\gamma} \equiv \left(A_{(p,\alpha)(q,\beta)}^{\gamma} \right) with\ A_{(p,\alpha)(q,\beta)}^{\gamma} \equiv \delta_{\alpha\gamma} \widehat{A}_{pq}^{\gamma} \delta_{\beta\gamma} \tag{17.7}$$

While the matrix $\underline{\underline{A}}^{t}: W \to W$ (*t* of *total*, not transposed) is defined by

$$\underline{\underline{A}}^{t} \equiv \sum_{\gamma=1}^{E} \underline{\underline{A}}^{\gamma} \tag{17.8}$$

Two fundamental properties of $\underline{\underline{A}}^{t}$ are:

$$\left(\underline{\underline{A}}^{t} \right)^{-1} = \sum_{\gamma=1}^{E} \left(\underline{\underline{A}}^{\gamma} \right)^{-1}\ and\ R^{-1} \underline{\underline{a}} \underline{\underline{A}}^{t} R = R^{-1} \underline{\underline{a}} \underline{\underline{A}}^{t} \underline{\underline{a}} R = \widehat{\underline{\underline{A}}} \tag{17.9}$$

The first of these equalities implies that, if the local Schur-complement, $\underline{\underline{S}}^{\alpha}$, of each $\underline{\underline{A}}^{\alpha}$ exists and is invertible, then

$$\underline{\underline{S}}^{t} \equiv \sum_{\alpha=1}^{E} \underline{\underline{S}}^{\alpha}\ and\ \left(\underline{\underline{S}}^{t} \right)^{-1} \equiv \sum_{\alpha=1}^{E} \left(\underline{\underline{S}}^{\alpha} \right)^{-1} \tag{17.10}$$

If we define $\underline{u}' \equiv R\underline{\widehat{u}}$, using the second equality of Eq.(17.9) it is seem that the *original problem* is equivalent to

$$\underline{\underline{a}} \underline{\underline{A}}^{t} \underline{u}' = \underline{\overline{f}}\ with\ \underline{\underline{j}} \underline{u}' = 0 \tag{17.11}$$

Once $\underline{u}' \in W$ has been obtained, we can recuperate $\underline{\widehat{u}} \in \widehat{W}$ by means of $\underline{\widehat{u}} = R^{-1} \underline{u}'$.

Let $\underline{\underline{a}}^{r}: W \to W_r$ be the orthogonal projection operator of W into W_r and observe that $\underline{\underline{a}} = \underline{\underline{a}} \underline{\underline{a}}^{r}$. Then, we define

$$\underline{\underline{A}} \equiv \underline{\underline{a}}^{r} \underline{\underline{A}}^{t} \underline{\underline{a}}^{r} \tag{17.12}$$

Next, we prove that a vector $\underline{u}' \in W_r$ fulfills Eq. (6.1), if and only if, it satisfies Eq. (17.11). Using the following property this result can be derived:

when $\underline{u}' \in W$ and $\underline{\underline{j}} \underline{u}' = 0$, the following relations are fulfilled,

$$\underline{u}' \in W_r\ and\ \underline{\underline{a}} \left(\underline{\underline{a}}^{r} \underline{\underline{A}}^{t} \underline{\underline{a}}^{r} \right) \underline{u}' = \underline{\underline{a}} \underline{\underline{A}}^{t} \underline{u}' \tag{17.13}$$

Appendix "B"
On the non-preconditioned Dirichlet-Dirichlet and Neumann-Neumann formulations

In order to place our developments in an adequate perspective we start by revisiting some elementary concepts on the formulations mentioned in the title of this Appendix. We also will draw from the Herrera's "*Theory of differential equations in discontinuous piecewise-defined functions*" [Herrera 2007]. The problem to be considered is: "*Find* $\in H^2(\Omega)$ *, such that:*

$$\left. \begin{array}{l} \mathcal{L}u = f_{\Omega}, in\ \Omega \\ u = 0\ \ on\ \partial\Omega \end{array} \right\} \tag{18.1}$$

Here, the standard notation for Sobolev spaces is being used. Under suitable conditions [Smith *et al*., 1996], the existence and uniqueness of solution of this problem is granted. This problem can also be formulated in a space of *discontinuous piecewise-defined functions* [Herrera 2007].

Let the domain Ω be decomposed into two subdomains Ω_1, and Ω_2, Figure 3. To be specific, it

will be assumed that the differential operator $\mathcal{L}$ is second order and we consider the space $H^2(\Omega_1) \oplus H^2(\Omega_2)$ of *discontinuous piecewise-defined functions* [Herrera 2007]. A function in such a space is defined independently in each one of the two subdomains and its restrictions to Ω_1 and Ω_2 belong to $H^2(\Omega_1)$ and $H^2(\Omega_2)$, respectively. Generally, it has non-zero jump discontinuities across the interface Γ, of the function itself and of its normal derivative. The notations for the *'jump'* and the *'average'*, across Γ, will be

$$[\![u]\!] = u_+ - u_- \; and \; \dot{\bar{u}} = \frac{1}{2}\left(u_+ + u_-\right), on\, \Gamma \tag{18.2}$$

respectively. Furthermore, the space $H^2(\Omega)$ is a subspace of $H^2(\Omega_1) \oplus H^2(\Omega_2)$. Indeed, let $u \in H^2(\Omega_1) \oplus H^2(\Omega_2)$, then $u \in H^2(\Omega)$ if, and only if,

$$[\![u]\!] = \left[\!\left[\frac{\partial u}{\partial n}\right]\!\right] = 0, on\, \Gamma \tag{18.3}$$

Therefore, a formulation of the problem of Eq. (18.1) is:

Find a $u \in H^2(\Omega_1) \oplus H^2(\Omega_2)$, such that:

$$\left.\begin{array}{l} \mathcal{L}u = f_\Omega, in\, \Omega \\ [\![u]\!] = 0 \;\; and \;\; \left[\!\left[\dfrac{\partial u}{\partial n}\right]\!\right] = 0 \; on\, \Gamma \\ u = 0 \;\; on\, \partial\Omega \end{array}\right\} \tag{18.4}$$

It should be observed that, when the values of the solution, u, are known on Γ, then u can be obtained, everywhere in Ω, by solving two Dirichlet boundary-value problems, one in each one of the subdomains Ω_1 and Ω_2; the title *Dirichlet-Dirichlet* for the procedure so obtained, stems from this fact. Similarly, when the values of the normal derivative, $\dfrac{\partial u}{\partial n}$, are known on Γ, then u can be obtained, everywhere in Ω, by solving two Neumann boundary-value problems, one in each one of the subdomains Ω_1 and Ω_2. These observations are the basis for the two approaches to domain decomposition methods that will be considered next: the Dirichlet-Dirichlet and the Neumann-Neumann methods.

The non-preeconditioned Dirichlet-Dirichlet problem

Let u_Γ be the restriction to Γ, of the solution u, then u is the unique solution of the following two Dirichlet problems:

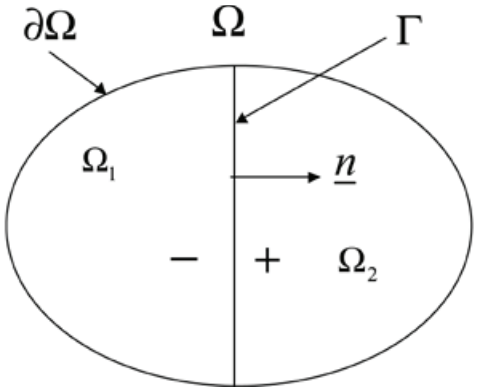

Figure 3. The *two-subdomain partition*

$$\left.\begin{array}{l} \mathcal{L}u = f_\Omega, in\, \Omega_\alpha, \alpha = 1,2 \\ u = u_\Gamma \;\; on\, \Gamma \\ u = 0 \;\; on\, \partial\Omega \end{array}\right\} \tag{18.5}$$

The Dirichlet-Dirichlet approach to domain decomposition consists in searching for the function u_Γ. Essentially, one chooses a sequence of trial-functions: $u_\Gamma^0, u_\Gamma^1, \ldots, u_\Gamma^n, \ldots$ until a satisfactory trial is found.

In this respect, a first question is: how to recognize when such a trial-function is satisfactory? We know that, when u_Γ is the restriction to Γ of the solution of our problem, the functions that are obtained by solving the two boundary-value problems of Eq. (18.5), fulfill the jump conditions indicated in Eq. (18.4). Now, in the case of the Dirichlet-Dirichlet approach, the jump of the function vanishes necessarily because in such a case:

$$[\![u]\!] = u_+ - u_- = u_\Gamma - u_\Gamma = 0 \tag{18.6}$$

However, generally, the condition

$$\left[\!\left[\frac{\partial u}{\partial n}\right]\!\right] = 0 \tag{18.7}$$

will not be satisfied. The choice of a trial u_Γ will be satisfactory if, only and only if, Eq. (18.7) is fulfilled.

Let us write the solution of the problem of Eq. (18.4) as

$$u = u_P + V \tag{18.8}$$

where

$$\left.\begin{array}{l} \mathcal{L}u_P = f_\Omega, in\, \Omega_\alpha, \alpha = 1,2 \\ u_P = 0 \;\; on\, \Gamma \\ u_P = 0 \;\; on\, \partial\Omega \end{array}\right\} \tag{18.9}$$

and, therefore,

$$\left.\begin{array}{l} \mathcal{L}v = 0, in\ \Omega_\alpha, \alpha = 1,2 \\ v = u_\Gamma \ \ on\ \Gamma \\ v = 0 \ \ on\ \partial\Omega \end{array}\right\} \qquad (18.10)$$

Hence, the choice of a trial u_Γ will be satisfactory if, only and only if,

$$\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] = -\left[\!\left[\frac{\partial u_P}{\partial n}\right]\!\right] \qquad (18.11)$$

From the point of view of the function $v \in H^2(\Omega_1) \oplus H^2(\Omega_2)$, we are searching for a function such that

$$\left.\begin{array}{l} \mathcal{L}v = 0, in\ \Omega_\alpha, \alpha = 1,2 \\ [\![v]\!] = 0, \ \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] = -\left[\!\left[\frac{\partial u_P}{\partial n}\right]\!\right], on\ \Gamma \\ v = 0 \ \ on\ \partial\Omega \end{array}\right\} \qquad (18.12)$$

This condition can be expressed by means of the Steklov-Poincaré operator, which for any function u_Γ defined on Γ, yields another function defined on Γ; namely [Quarteroni and Valli 1999]:

$$\tau(u_\Gamma) \equiv \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right], on\ \Gamma \qquad (18.13)$$

where v fulfills Eq. (18.10). Then Eq. (18.12) is equivalent to:

$$\tau(u_\Gamma) = -\left[\!\left[\frac{\partial u_P}{\partial n}\right]\!\right], on\ \Gamma \qquad (18.14)$$

In the *DVS-framework*, the discrete version of the Steklov-Poincaré operator is $\underline{\underline{\underline{\underline{aS}}}}$ (see Section 9 of [Herrera and Yates 2010]). Therefore, the discrete version of Eq. (18.14) is

$$\underline{\underline{\underline{\underline{aS}}}}\,\underline{v} = -\underline{\underline{\underline{\underline{aS}}}}\,\underline{u}_P \ together\ with\ \underline{\underline{j}}\,\underline{v} = 0 \qquad (18.15)$$

This equation can be reconciled with Eq.(7.3) making the replacements $-\left[\!\left[\frac{\partial u_P}{\partial n}\right]\!\right] \leftrightarrow \underline{f}_\Delta$ and $\underline{v} \leftrightarrow \underline{u}_\Delta$.

The non-preconditioned Neumann-Neumann problem

Again, we write u for the solution of Eq. (18.4). Let be $q_\Gamma \equiv \frac{\partial u}{\partial n}$, on Γ, then u is the unique solution of the following two Neumann problems:

$$\left.\begin{array}{l} \mathcal{L}u = f_\Omega, in\ \Omega_\alpha, \alpha = 1,2 \\ \frac{\partial u}{\partial n} = q_\Gamma \ \ on\ \Gamma \\ u = 0 \ \ on\ \partial\Omega \end{array}\right\} \qquad (18.16)$$

The title Neumann-Neumann approach comes from the fact that Eq. (18.16) implies solving a Neumann in Ω_1 and another Neumann problem in Ω_2. This approach consists in searching for the function q_Γ. Independently of the value of q_Γ, any solution of Eq. (18.16) satisfies:

$$\left[\!\left[\frac{\partial u}{\partial n}\right]\!\right] = \left(\frac{\partial u}{\partial n}\right)_+ - \left(\frac{\partial u}{\partial n}\right)_- = q_\Gamma - q_\Gamma = 0 \qquad (18.17)$$

However, generally

$$[\![u]\!] \neq 0 \qquad (18.18)$$

We write again the sought solution as in Eq. (18.8), but replace Eqs. (18.9) and (18.10) by

$$\left.\begin{array}{l} \mathcal{L}u_P = f_\Omega, in\ \Omega_\alpha, \alpha = 1,2 \\ \frac{\partial u_P}{\partial n} = 0 \ \ on\ \Gamma \\ u_P = 0 \ \ on\ \partial\Omega \end{array}\right\} \qquad (18.19)$$

and, therefore

$$\left.\begin{array}{l} \mathcal{L}v = 0, in\ \Omega_\alpha, \alpha = 1,2 \\ \frac{\partial v}{\partial n} = q_\Gamma \ \ on\ \Gamma \\ v = 0 \ \ on\ \partial\Omega \end{array}\right\} \qquad (18.20)$$

Hence, the function $v \in H^2(\Omega_1) \oplus H^2(\Omega_2)$ is characterized by:

$$\left.\begin{array}{l} \mathcal{L}v = 0, in\ \Omega_\alpha, \alpha = 1,2 \\ [\![v]\!] = -[\![u_P]\!], \ \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] = 0, on\ \Gamma \\ v = 0 \ \ on\ \partial\Omega \end{array}\right\} \qquad (18.21)$$

For the Neumann-Neumann formulation there is a counter-part of the Steklov-Poincaré operator: Given any function q_Γ, defined on Γ, we define $\mu(q_\Gamma)$ to be a function on Γ given by:

$$\mu\left(q_\Gamma\right) \equiv \left[\!\left[v \right]\!\right], on\, \Gamma \qquad (18.22)$$

Here *v* satisfies Eq. (18.20). Then, Eq. (18.21) is equivalent to

$$\mu\left(q_\Gamma\right) = -\left[\!\left[u_P \right]\!\right], on\, \Gamma \qquad (18.23)$$

To obtain the discrete version of the operator μ, we first write the discrete version of Eq. (18.20); it is:

$$\underline{\underline{j}}\,\underline{\underline{S}}\,\underline{v} = \underline{q}_\Gamma \; together\; with\; \underline{\underline{a}}\,\underline{\underline{S}}\,\underline{v} = 0 \qquad (18.24)$$

Therefore, while

$$\underline{\underline{a}}\,\underline{q}_\Gamma = 0 \; and \;\; \underline{\underline{j}}\,\underline{v} = \underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{\underline{S}}\,\underline{v} = \underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{\underline{j}}\,\underline{\underline{S}}\,\underline{v} = \underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{q}_\Gamma \qquad (18.25)$$

This establishes the correspondence $\mu \leftrightarrow \underline{\underline{j}}\,\underline{\underline{S}}^{-1}$ The discrete version of Eq. (18.23), is

$$\underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{q}_\Gamma = -\underline{\underline{j}}\,\underline{u}_P \; together\; with\; \underline{\underline{a}}\,\underline{q}_\Gamma = 0 \qquad (18.26)$$

Another option is to tackle the discrete version of Eq. (18.21), without resource to the operator μ, counter-part of the Steklov-Poincaré operator. The corresponding problem is

$$\underline{\underline{j}}\,\underline{v} = -\underline{\underline{j}}\,\underline{u}_P \; and\; \underline{\underline{a}}\,\underline{\underline{S}}\,\underline{v} = 0 \qquad (18.27)$$

However, it should be observed that Eq. (18.27) does not define an iterative algorithm because $\underline{\underline{a}}\,\underline{\underline{S}}\,\underline{\underline{j}} \neq 0$. An equation equivalent to it, but which can be applied iteratively is

$$\underline{\underline{S}}^{-1}\,\underline{\underline{j}}\,\underline{v} = -\underline{\underline{S}}^{-1}\,\underline{\underline{j}}\,\underline{u}_P \; and\; \underline{\underline{a}}\,\underline{\underline{S}}\,\underline{v} = 0 \qquad (18.28)$$

Appendix "C"
The Lagrange multipliers formulation

To obtain the Lagrange multipliers formulation, we write

$$\left.\begin{array}{c} J\left(\underline{u}\right) \equiv \dfrac{1}{2}\,\underline{u}\cdot\underline{\underline{S}}\,\underline{u} - \underline{u}\cdot\underline{f} \rightarrow \min \\ \underline{\underline{j}}\,\underline{u} = 0 \end{array}\right\} \qquad (19.1)$$

Taking the variation in Eq. (19.1), it is obtained

$$\underline{\underline{S}}\,\underline{u} + \underline{\underline{j}}\,\underline{\lambda} = \underline{f} \; together\; with\; \underline{\underline{j}}\,\underline{u} = 0 \qquad (19.2)$$

To assure uniqueness for $\underline{\lambda}$, we impose the condition $\underline{\underline{a}}\,\underline{\lambda} = 0$, so that $\underline{\underline{j}}\,\underline{\lambda} = \underline{\lambda}$. Then, Eq. (19.2) implies

$$\underline{\underline{a}}\,\underline{\underline{S}}\,\underline{u} + \underline{\underline{j}}\,\underline{\underline{S}}\,\underline{u} + \underline{\lambda} = \underline{f} \qquad (19.3)$$

Multiplying by $\underline{\underline{j}}$ and $\underline{\underline{a}}$ successively, it is obtained

$$\underline{\lambda} = -\underline{\underline{j}}\,\underline{\underline{S}}\,\underline{u} \;\; and\; \underline{\underline{a}}\,\underline{\underline{S}}\,\underline{u} = \underline{f} \qquad (19.4)$$

Then, Eq. (8.3) is clear.