Unified Multipliers-Free Theory of Dual-Primal Domain Decomposition Methods

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The concept of dual-primal methods can be formulated in a manner that incorporates, as a subclass, the non preconditioned case. Using such a generalized concept, in this article without recourse to "Lagrange multipliers," we introduce an all-inclusive unified theory of nonoverlapping domain decomposition methods (DDMs). One-level methods, such as Schur-complement and one-level FETI, as well as two-level methods, such as Neumann-Neumann and preconditioned FETI, are incorporated in a unified manner. Different choices of the dual subspaces yield the different dual-primal preconditioners reported in the literature. In this unified theory, the procedures are carried out directly on the matrices, independently of the differential equations that originated them. This feature reduces considerably the code-development effort required for their implementation and permit, for example, transforming 2D codes into 3D codes easily. Another source of this simplification is the introduction of two projection-matrices, generalizations of the average and jump of a function, which possess superior computational properties. In particular, on the basis of numerical results reported there, we claim that our jump matrix is the optimal choice of the $\boldsymbol{B}$ operator of the FETI methods. A new formula for the Steklov-Poincaré operator, at the discrete level, is also introduced. Numer Methods Partial Differential Eq 000: 000–000, 2008

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I. INTRODUCTION

Mathematical models of many systems of interest, including very important continuous systems of Engineering and Science, lead to a great variety of partial differential equations whose solution methods are based on the computational processing of large-scale algebraic systems. Furthermore, the incredible growth experienced by the existing computational hardware and software has made amenable to effective treatment an ever increasing diversity and complexity of problems, posed by engineering and scientific applications.

Parallel computing is outstanding among the new computational tools, especially at present when further increases in hardware speed apparently have reached insurmountable barriers [1].

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Therefore, the emergence of parallel computing during the last 20 years or so has prompted a continued and systematic effort on the part of the computational-modeling community with the purpose of harnessing it for the endeavor of solving partial differential equations. Very early after such an effort began, it was recognized that domain decomposition methods (DDM) were the most effective technique for applying parallel computing to the solution of partial differential equations, since such an approach drastically simplifies the coordination of many processors that carry out the different tasks and also significantly reduces the requirements of information-transmission between them. Up to this date, the DDM organization has held 18 international conferences and their proceedings constitute a valuable source of information [2]. Since research work on DDM has been abundant, survey articles and monographs are especially useful and some of the most distinguished of them are included in the Refs. 3–11. We will draw especially from [11] due to its broadness, extensive coverage of material pertinent to the subject of this article, and its recent publication date.

There are many approaches to DDM, albeit lately much of the effort has gone into iterative substructuring methods in nonoverlapping partitions such as Neumann-Neumann, Dirichlet-Dirichlet (preconditioned FETI), and FETI [11]. In the processing of such methods, discontinuous functions are generated at some stages and so a conspicuous feature of them is that, after a domain partition has been introduced, they use discontinuous piecewise-defined functions as base functions for representing the approximate solutions of the partial differential equations (see, for example [9, 11]). Until recently, the treatment of discontinuous functions had been based on the use of Lagrange multipliers (see [12], for a review of this topic). However, a more direct approach is feasible, as has been shown by Herrera in two very recent papers in which discontinuous functions are dealt-with without recourse to Lagrange multipliers [12, 13]. Significant improvements and generalizations of such an approach have been made during the last year or so, and this article is devoted to report that progress.

As indicated by its title, this article summarizes contributions in a "Unified Multipliers-Free Theory of Nonoverlapping Iterative Domain Decompositions Methods." In this theory, the domain decomposition procedures are carried out directly on the matrices independently of the partial differential equations that originated them; so, throughout the article, function-spaces are not discussed. The methods so derived possess many advantages; therefore, for example, codes developed for 2D-problems can be easily modified for its application to 3D-problems.

First, in this theory a framework is established with the purpose of unifying different approaches. In it, the usual concepts of primal and dual nodes are slightly modified: a node is primal if and only if it is not divided, and dual nodes are all others. This permits incorporating dual-primal preconditioners from the start and, therefore, all the methods and, more importantly, matrices derived in this setting can, in a direct manner, be applied indistinctly with or without dual-primal preconditioners. Different choices of the dual subspaces yield the different dual-primal preconditioners reported in the literature. Second, in that setting and without recourse to Lagrange multipliers, a DDM matrix-formulation of problems in discontinuous functions is introduced. The results so obtained constitute a unified matrix and algorithmic formulation of dual-primal methods, which incorporates as a particular case that for which a node is divided if and only if it lies on the interior boundary; i.e., all interior nodes are primal and all internal-boundary nodes are dual. This yields an all-inclusive theory of one-level nonoverlapping DDMs that, in particular, incorporates the *Schur* complement method (obtained using a Dirichlet approach) and a one-level *FETI* method (obtained using a Neumann approach). Then, a general matrix-formulation is constructed for two-level nonoverlapping DDMs, which is based on a scheme of wide applicability here called the "*round-trip algorithm*", and that was first introduced by Herrera in [13]. In the

developments presented in this article, such DDMs consist of the solution of a Dirichlet problem followed by a Neumann problem, or the same but in reverse order; so, they essentially are Neumann-Neumann and preconditioned FETI methods [11], respectively, except that they do not use Lagrange multipliers and, furthermore, they ab initio incorporate dual-primal preconditioners and yield explicit matrix expressions valid when they are applied. This feature considerably reduces the code-development effort required for its implementation, as it is further discussed in Section XII, devoted to numerical results.

Two projection-matrices, $\underline{\underline{a}}$ and $\underline{\underline{j}}$ are introduced, which are generalizations of the *"average"* and *"jump"* of a function and can be effectively applied at the discrete level (i.e., to vectors) not only at internal-boundary nodes but at edges and corners, as well. The matrix $\underline{\underline{a}}$ yields, when applied to any vector, the projection on the subspace of continuous vectors, whereas $\underline{\underline{j}}$ is the projection on its orthogonal complement, so that $\underline{\underline{a}}\,\underline{\underline{j}} = \underline{\underline{j}}\,\underline{\underline{a}} = \underline{\underline{0}}$. Furthermore, these matrices, being, symmetric, nonnegative and orthogonal projection-matrices, have superior computational properties and, on the basis of numerical results here reported, we claim that the $\underline{\underline{j}}$ operator is the optimal choice for the $\boldsymbol{B}$ operator of the FETI methods [11], as discussed in the section on Computational Results. Construction of these matrices is very simple; indeed, $\underline{\underline{a}}$ is the average over each node and, once $\underline{\underline{a}}$ is available, $\underline{\underline{j}}$ easily derives from it.

Numerical experiments that use dual-primal competitive preconditioners, which incorporate a coarse space, were also carried out as part of the research supporting this study. Furthermore, in Section XI an efficient method of solution applicable to general preconditioned dual-primal formulations is introduced. The unified theory presented here introduces certain number of new concepts and revises some others; in this respect, it should be mentioned that the framework in which the theory is based implies a slight modification of the expression for *Steklov-Poincaré* operator at the discrete level (see the end of Section III). A significant difference of the new expression here proposed for it, with respect to standard formulas, lies in the fact that the new formula of ours does not contain the right-hand side of the equation to be solved; winning thereby, in theoretical consistency.

The theory and algorithms derived without recourse to Lagrange multipliers will be referred to as multipliers-free. Other unified theories have been published [14, 15]. We believe our unified theory goes beyond such contributions as follows: it is more systematic, complete, and covers the specifics more thoroughly.

II. DUAL-PRIMAL SPACES

Let $\bar{\Omega}$ denote a discrete set, whose members will be referred to as "degrees of freedom." In this setting "nodes" are subsets of $\bar{\Omega}$, to be denoted by $\mathrm{Z} \subset \bar{\Omega}$. Furthermore, the collection of the nodes constitute a partition, $\mathcal{P}$, of $\bar{\Omega}$; i.e.,

$$\begin{cases} \bar{\Omega} = \bigcup_{\mathrm{Z}\in P} \mathrm{Z} \\ \mathrm{Z} \cap \Upsilon = \phi, \text{ when } \mathrm{Z} \neq \Upsilon \text{ and } \mathrm{Z}, \Upsilon \in \mathcal{P} \end{cases} \tag{2.1}$$

Nodes with cardinality one are said to be "primal" while those whose cardinality is greater than one, are said to be "dual nodes." The sets $\Pi \subset \bar{\Omega}$ and $\Delta \subset \bar{\Omega}$ are defined as the union of all the degrees of freedom associated with primal nodes and as the union of all the degrees of freedom associated with dual nodes, respectively, so that

$$\bar{\Omega} = \Pi \cup \Delta \text{ and } \phi = \Pi \cap \Delta \tag{2.2}$$

Notice that every real-valued function defined in $\bar{\Omega}$ is a vector (so, they will be referred to indistinctly as functions or vectors). The set of such vectors, of dimension "d", equal to the cardinality of $\bar{\Omega}$, constitutes a linear space that will be denoted by $\tilde{D}(\bar{\Omega})$. When $\underline{u} \in \tilde{D}(\bar{\Omega})$, we write

$$\underline{u} \equiv (u_1, \ldots, u_d) \tag{2.3}$$

Then, the "Euclidean inner product", which is the only one to be considered in this section, is defined to be

$$\underline{u} \bullet \underline{w} \equiv \sum_{i \in \bar{\Omega}} u_i w_i = \sum_{i=1}^{d} u_i w_i \tag{2.4}$$

Correspondingly, $\tilde{D}(\mathrm{Z}) \subset \tilde{D}(\bar{\Omega})$, $\tilde{D}(\Pi) \subset \tilde{D}(\bar{\Omega})$, and $\tilde{D}(\Delta) \subset \tilde{D}(\bar{\Omega})$ will be the linear spaces whose elements are the real-valued functions defined in Z, Π, and Δ, where Z is any node, respectively. More properly, $\tilde{D}(\mathrm{Z})$, $\tilde{D}(\Pi)$, and $\tilde{D}(\Delta)$ will be the linear subspaces of $\tilde{D}(\bar{\Omega})$ whose vectors vanish identically outside of Z, Π, and Δ, respectively. Then

$$\tilde{D}(\bar{\Omega}) = \tilde{D}(\Pi) \oplus \tilde{D}(\Delta) \tag{2.5}$$

Here, and in what follows, the symbol $\oplus$ stands for the direct sum of two linear spaces; thus, Eq. (2.5) is fulfilled if and only if

$$\begin{cases} \tilde{D}(\bar{\Omega}) = \tilde{D}(\Pi) + \tilde{D}(\Delta) \\ \{0\} = \tilde{D}(\Pi) \cap \tilde{D}(\Delta) \end{cases} \tag{2.6}$$

Therefore, vectors of $\tilde{D}(\bar{\Omega})$ can be represented in a unique manner as

$$\underline{u} = (\underline{u}_\Pi, \underline{u}_\Delta) = \underline{u}_\Pi + \underline{u}_\Delta, \text{ with } \underline{u}_\Pi \in \tilde{D}(\Pi) \text{ and } \underline{u}_\Delta \in \tilde{D}(\Delta) \tag{2.7}$$

A vector $\underline{u} \in \tilde{D}(\bar{\Omega})$ is said to be continuous when for every node Z, one has

$$u_i = u_j, \quad \forall i, j \in \mathrm{Z} \tag{2.8}$$

That is, at every node $\mathrm{Z} \in \mathcal{P}$ the value of such a vector is unique. Continuous vectors constitute a linear subspace of $\tilde{D}(\bar{\Omega})$ that will be denoted by $\bar{D}(\bar{\Omega})$. We notice that

$$\tilde{D}(\Pi) \subset \bar{D}(\bar{\Omega}) \tag{2.9}$$

Two matrices $\underline{\underline{a}} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ and $\underline{\underline{j}} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ are now introduced, which are defined by

$$\underline{\underline{a}}\underline{u} = Proj_{\bar{D}}\underline{u} \text{ and } \underline{\underline{j}} = \underline{\underline{I}} - \underline{\underline{a}} \tag{2.10}$$

Here, $\underline{\underline{I}}$ is the identity matrix and the projection on $\bar{D}$ is taken with respect to the *Euclidean* inner product. In view of this definition we have

$$\bar{D}(\bar{\Omega}) \equiv \underline{\underline{a}}\tilde{D}(\bar{\Omega}) \tag{2.11}$$

An obvious and important property is that

$$\underline{\underline{I}} = \underline{\underline{a}} + \underline{\underline{j}} \tag{2.12}$$

Furthermore, $\underline{\underline{j}}$ is also a projection; indeed, it is the projection on the orthogonal complement of $\bar{D}$. Therefore, $\underline{\underline{a}}$ and $\underline{\underline{j}}$ are both symmetric, nonnegative, and idempotent. We also notice that

$$\underline{\underline{a}}\,\underline{\underline{j}} = \underline{\underline{j}}\,\underline{\underline{a}} = \underline{\underline{0}} \tag{2.13}$$

In particular

$$\underline{\underline{j}}\,\bar{D}(\bar{\Omega}) = \{0\} \tag{2.14}$$

The construction of the matrix $\underline{\underline{a}}$ is straightforward: "given a vector of $\underline{u} \in \tilde{D}(\bar{\Omega})$ to obtain $\underline{\underline{a}}\,\underline{u}$, at every degree of freedom belonging to a node the value of $\underline{\underline{a}}\,\underline{u}$ equals the average of $\underline{u}$ over that node", i.e.,

$$(\underline{\underline{a}}\,\underline{u})_i = \frac{1}{|\mathrm{Z}|}\sum_{j\in\mathbf{Z}} u_j, \text{ whenever } i \in \mathrm{Z} \tag{2.15}$$

Here, |Z| is the cardinality of Z. As for $\underline{\underline{j}}$, it does not need to be computed since

$$\underline{\underline{j}}\,\underline{u} = \underline{u} - \underline{\underline{a}}\,\underline{u}, \quad \forall \underline{u} \in \tilde{D}(\bar{\Omega}) \tag{2.16}$$

On the other hand,

$$\underline{\underline{a}}\,\underline{u} = \underline{u} \text{ and } \underline{\underline{j}}\,\underline{u} = 0, \quad \forall \underline{u} \in \tilde{D}(\Pi) \tag{2.17}$$

It can also be seen that, for every node $\mathrm{Z} \in \mathcal{P}$:

$$\underline{\underline{a}}\,\underline{u} \in \tilde{D}(\mathrm{Z}) \text{ and } \underline{\underline{j}}\,\underline{u} \in \tilde{D}(\mathrm{Z}), \text{ when } \underline{u} \in \tilde{D}(\mathrm{Z}) \tag{2.18}$$

Then, it is clear that

$$\underline{\underline{a}}\{\tilde{D}(\Pi)\} \subseteq \tilde{D}(\Pi), \underline{\underline{a}}\{\tilde{D}(\Delta)\} \subset \tilde{D}(\Delta) \text{ and } \underline{\underline{j}}\{\tilde{D}(\Pi)\} = \{0\} \tag{2.19}$$

Using the notation of Eq. (2.7), one has $\underline{\underline{j}}\,\tilde{\underline{u}} = \underline{\underline{j}}\,\tilde{\underline{u}}_\Delta \in \tilde{D}(\Delta)$ for every $\tilde{\underline{u}} \in \tilde{D}(\bar{\Omega})$; therefore,

$$\underline{\underline{j}}\{\tilde{D}(\bar{\Omega})\} = \underline{\underline{j}}\{\tilde{D}(\Delta)\} \subset \tilde{D}(\Delta) \tag{2.20}$$

A more formal treatment of the matrices $\underline{\underline{a}}$ and $\underline{\underline{j}}$, as well as some additional details, is given in the Appendix.

Now, we define

$$\begin{aligned} \tilde{D}_1(\Delta) &\equiv \underline{\underline{j}}\,\tilde{D}(\Delta) = \underline{\underline{j}}\,\tilde{D}(\bar{\Omega}) \\ \tilde{D}_2(\Delta) &\equiv \underline{\underline{a}}\,\tilde{D}(\Delta) \end{aligned} \tag{2.21}$$

The following properties of these subspaces of $\tilde{D}(\Delta)$ are listed:

- $\tilde{D}_1(\Delta)$ is the orthogonal complement of $\bar{D}(\bar{\Omega}) = \underline{\underline{a}}\tilde{D}(\bar{\Omega})$;
- $$\tilde{D}(\bar{\Omega}) = \tilde{D}_1(\Delta) \oplus \bar{D}(\bar{\Omega}) \tag{2.22}$$
- $$\tilde{D}(\Delta) = \tilde{D}_1(\Delta) \oplus \tilde{D}_2(\Delta) \tag{2.23}$$
- $\tilde{D}_1(\Delta)$ and $\tilde{D}_2(\Delta)$ are orthogonal complements relative to $\tilde{D}(\Delta)$.
- $$\bar{D}(\bar{\Omega}) = \tilde{D}_2(\Delta) \oplus \tilde{D}(\Pi) \tag{2.24}$$
$$\tilde{D}(\bar{\Omega}) = \underline{\underline{a}}\tilde{D}(\bar{\Omega}) \oplus \underline{\underline{j}}\tilde{D}(\bar{\Omega}) = \bar{D}(\bar{\Omega}) \oplus \underline{\underline{j}}\tilde{D}(\bar{\Omega}) = \tilde{D}(\Pi) \oplus \tilde{D}_1(\Delta) \oplus \tilde{D}_2(\Delta) \tag{2.25}$$

Other properties implied by the above results are

$$\begin{aligned}
\bar{D}(\bar{\Omega}) &= \left\{ \underline{u} \in \tilde{D}(\bar{\Omega}) \middle| \underline{\underline{j}}\underline{u} = 0 \right\} = \left\{ \underline{u} \in \tilde{D}(\bar{\Omega}) \middle| \underline{\underline{a}}\underline{u} = \underline{u} \right\} \\
\tilde{D}_1(\Delta) &= \left\{ \underline{u} \in \tilde{D}(\bar{\Omega}) \middle| \underline{\underline{a}}\underline{u} = 0 \right\} = \left\{ \underline{u} \in \tilde{D}(\bar{\Omega}) \middle| \underline{\underline{j}}\underline{u} = \underline{u} \right\} \\
\tilde{D}_2(\Delta) &= \left\{ \underline{u} \in \tilde{D}(\Delta) \middle| \underline{\underline{j}}\underline{u} = 0 \right\} = \left\{ \underline{u} \in \tilde{D}(\Delta) \middle| \underline{\underline{a}}\underline{u} = \underline{u} \right\}
\end{aligned} \tag{2.26}$$

These relations, together with Eq. (2.9), will also be used in the sequel.

Other notation, which will also be used, is that for every function $\underline{u} \in \tilde{D}(\bar{\Omega})$, we will write

$$\overset{\bullet}{\underline{\hat{u}}} \equiv \underline{\underline{a}}\underline{u} \text{ and } [\![\underline{u}]\!] \equiv \underline{\underline{j}}\underline{u} \tag{2.27}$$

Then $\overset{\bullet}{\underline{\hat{u}}} \in \bar{D}(\bar{\Omega})$, whereas $[\![\underline{u}]\!]$ belongs to $\tilde{D}_1(\Delta) \subset \tilde{D}(\Delta)$. It should be noticed that, in view of Eq. (2.25), any $\underline{u} \in \tilde{D}(\bar{\Omega})$ can be written uniquely as

$$\underline{u} = \underline{u}_\Pi + \underline{u}_\Delta = \underline{u}_\Pi + \underline{u}_{\Delta 1} + \underline{u}_{\Delta 2} \text{ with } \underline{u}_\Pi \in \tilde{D}(\Pi), \underline{u}_{\Delta 1} \in \tilde{D}_1(\Delta) \text{ and } \underline{u}_{\Delta 2} \in \tilde{D}_2(\Delta) \tag{2.28}$$

with $\underline{u}_{\Delta 1} \equiv [\![\underline{u}_\Delta]\!]$, $\underline{u}_{\Delta 2} \equiv \overset{\bullet}{\underline{\hat{u}}}_\Delta$ and $\underline{u}_\Delta = \underline{u}_{\Delta 1} + \underline{u}_{\Delta 2}$.

III. GREEN-HERRERA FORMULA FOR MATRICES AND THE STEKLOV-POINCARÉ OPERATOR

Now, let $\underline{\underline{A}} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ be a symmetric and positive definite matrix. The "energy inner product" is defined by

$$(\underline{u}, \underline{w}) \equiv \underline{u} \bullet \underline{\underline{A}}\underline{w}, \ \forall \underline{u}, \underline{w} \in \tilde{D}(\bar{\Omega}) \tag{3.1}$$

The linear space, $\tilde{D}(\bar{\Omega})$ becomes a (finite-dimensional) Hilbert space when it is provided with the energy inner product. We will write

$$\underline{\underline{A}} \equiv \begin{pmatrix} \underline{\underline{A}}_{\Pi\Pi} & \underline{\underline{A}}_{\Pi\Delta} \\ \underline{\underline{A}}_{\Delta\Pi} & \underline{\underline{A}}_{\Delta\Delta} \end{pmatrix} \tag{3.2}$$

The notation here is such that

$$\begin{cases} \underline{\underline{A}}_{\Pi\Pi} : \tilde{D}(\Pi) \to \tilde{D}(\Pi), & \underline{\underline{A}}_{\Pi\Delta} : \tilde{D}(\Delta) \to \tilde{D}(\Pi) \\ \underline{\underline{A}}_{\Delta\Pi} : \tilde{D}(\Pi) \to \tilde{D}(\Delta), & \underline{\underline{A}}_{\Delta\Delta} : \tilde{D}(\Delta) \to \tilde{D}(\Delta) \end{cases} \tag{3.3}$$

We introduce the following definitions

$$\underline{\underline{L}} \equiv \begin{pmatrix} \underline{\underline{A}}_{\Pi\Pi} & \underline{\underline{A}}_{\Pi\Delta} \\ 0 & 0 \end{pmatrix} \text{ and } \underline{\underline{R}} \equiv \begin{pmatrix} 0 & 0 \\ \underline{\underline{A}}_{\Delta\Pi} & \underline{\underline{A}}_{\Delta\Delta} \end{pmatrix} \tag{3.4}$$

The matrices $\left[\!\left[\underline{\underline{R}}\right]\!\right] : \tilde{D}(\Omega) \to \tilde{D}(\Omega)$ and $\dot{\underline{\underline{\widehat{R}}}} : \tilde{D}(\Omega) \to \tilde{D}(\Omega)$ are defined by

$$\left[\!\left[\underline{\underline{R}}\right]\!\right] \equiv \underline{\underline{a}}\,\underline{\underline{R}} \text{ and } \dot{\underline{\underline{\widehat{R}}}} \equiv \underline{\underline{j}}\,\underline{\underline{R}} \tag{3.5}$$

When Eq. (3.5) holds, in view of Eq.(2.8), one has

$$\underline{\underline{R}} = \dot{\underline{\underline{\widehat{R}}}} + \left[\!\left[\underline{\underline{R}}\right]\!\right] \tag{3.6}$$

Furthermore

$$\left[\!\left[\underline{\underline{R}}\right]\!\right]^T \equiv \underline{\underline{R}}^T \underline{\underline{a}} \text{and} \left(\dot{\underline{\underline{\widehat{R}}}}\right)^T \equiv \underline{\underline{R}}^T \underline{\underline{j}} \tag{3.7}$$

We notice that the ranges of $\underline{\underline{L}}$ and $\underline{\underline{R}}$ are $\tilde{D}(\Pi)$ and $\tilde{D}(\Delta)$, respectively, whereas those of $\left[\!\left[\underline{\underline{R}}\right]\!\right]$ and $\dot{\underline{\underline{\widehat{R}}}}$ are contained in $\tilde{D}(\Delta)$. Even more, these latter two ranges are linearly independent.

With this notation, the following relations are satisfied:

$$\underline{\underline{A}} = \underline{\underline{L}} + \underline{\underline{R}} \tag{3.8}$$

$$\underline{w} \bullet \underline{\underline{L}}\,\underline{u} - \underline{u} \bullet \underline{\underline{L}}\,\underline{w} = \underline{u} \bullet \underline{\underline{R}}\,\underline{w} - \underline{w} \bullet \underline{\underline{R}}\,\underline{u}, \forall \underline{u}, \underline{w} \in \tilde{D}(\bar{\Omega}) \tag{3.9}$$

Furthermore, Eq. (3.9) implies

$$\underline{w} \bullet \underline{\underline{L}}\,\underline{u} - [\![\underline{u}]\!] \bullet \dot{\underline{\underline{\widehat{R}}}}\,\underline{w} + \dot{\underline{\widehat{w}}} \bullet \left[\!\left[\underline{\underline{R}}\right]\!\right] \underline{u} = \underline{u} \bullet \underline{\underline{L}}\,\underline{w} - [\![\underline{w}]\!] \bullet \dot{\underline{\underline{\widehat{R}}}}\,\underline{u} + \dot{\underline{\widehat{u}}} \bullet \left[\!\left[\underline{\underline{R}}\right]\!\right] \underline{w},$$

$$\forall \underline{u}, \underline{w} \in \tilde{D}(\bar{\Omega}) \tag{3.10}$$

Eq. (3.10), will be referred to as the "Green-Herrera formula for matrices". It establishes that the matrix $\underline{\underline{L}} - \dot{\underline{\underline{\widehat{R}}}}^T + \left[\!\left[\underline{\underline{R}}\right]\!\right]$ is symmetric; i.e.,

$$\underline{\underline{L}} - \dot{\underline{\underline{\widehat{R}}}}^T + \left[\!\left[\underline{\underline{R}}\right]\!\right] = \left(\underline{\underline{L}} - \dot{\underline{\underline{\widehat{R}}}}^T + \left[\!\left[\underline{\underline{R}}\right]\!\right]\right)^T \tag{3.11}$$

Hence,

$$\underline{\underline{L}} - \dot{\underline{\underline{\widehat{R}}}}^T + \left[\!\left[\underline{\underline{R}}\right]\!\right] = \underline{\underline{L}}^T - \dot{\underline{\underline{\widehat{R}}}} + \left[\!\left[\underline{\underline{R}}\right]\!\right]^T \tag{3.12}$$

or, equivalently:

$$\underline{\underline{L}} - \underline{\underline{R}}^T \underline{j} + \underline{a}\underline{\underline{R}} = \underline{\underline{L}}^T - \underline{j}\underline{\underline{R}} + \underline{\underline{R}}^T \underline{a} \tag{3.13}$$

In the light of Eq. (3.10), it is now possible to explain the motivation of the definitions introduced by Eq. (3.5). The Green-Herrera formulae were originally introduced in a couple of papers [16, 17] and have been used as basis for several numerical methods such as the Eulerian-Lagrangian localized adjoint method (ELLAM) [18], Trefftz method [19], and FEM with optimal functions (FEM-OF [20]). For Laplace's differential operator acting on discontinuous piecewise–defined functions that satisfy homogeneous boundary conditions, Green-Herrera formula is (see, for example [19, 20]):

$$\int_\Omega w\mathcal{L}u dx + \int_\Gamma \left\{ [\![u]\!] \widehat{\frac{\partial w}{\partial n}} - \dot{\widehat{w}} \left[\!\left[\frac{\partial u}{\partial n} \right]\!\right] \right\} dx = \int_\Omega u\mathcal{L}w dx + \int_\Gamma \left\{ [\![w]\!] \dot{\widehat{\frac{\partial u}{\partial n}}} - \dot{\widehat{u}} \left[\!\left[\frac{\partial w}{\partial n} \right]\!\right] \right\} dx \tag{3.14}$$

Comparison of Eqs. (3.10) and (3.14) yields the following correspondences:

$$\left.\begin{array}{l} \mathcal{L} \leftrightarrow \underline{\underline{L}} \\ [\![u]\!] \leftrightarrow \left[\!\left[\underline{u}\right]\!\right] \\ \dot{\widehat{u}} \leftrightarrow \dot{\widehat{\underline{u}}} \end{array}\right\} \text{ and } \left\{\begin{array}{l} \left[\!\left[\frac{\partial u}{\partial n} \right]\!\right] \leftrightarrow -\left[\!\left[\underline{\underline{R}} \right]\!\right] \underline{u} \\ \dot{\widehat{\frac{\partial u}{\partial n}}} \leftrightarrow -\dot{\widehat{\underline{\underline{R}}}}\, \underline{u} \end{array}\right. \tag{3.15}$$

Of course, we could have modified the definitions of Eq. (3.5), by changing the sign there, so that $\left[\!\left[\underline{\underline{R}} \right]\!\right] \underline{u}$ and $\dot{\widehat{\underline{\underline{R}}}}\, \underline{u}$ would correspond directly to the jump and the average of the normal derivative, respectively. However, we refrained from doing so since that would lead to unnecessary notation complications in our developments.

Another example is the Green-Herrera formula for the general elliptic operator, symmetric, and second order, which is

$$\int_\Omega w\mathcal{L}u dx + \int_\Gamma \left\{ [\![u]\!] \dot{\widehat{\underline{a}_n \bullet \nabla w}} - \dot{\widehat{w}} \left[\!\left[\underline{a}_n \bullet \nabla u \right]\!\right] \right\} dx$$
$$= \int_\Omega u\mathcal{L}w dx + \int_\Gamma \left\{ [\![w]\!] \dot{\widehat{\underline{a}_n \bullet \nabla u}} - \dot{\widehat{u}} \left[\!\left[\underline{a}_n \bullet \nabla w \right]\!\right] \right\} dx \tag{3.16}$$

The correspondence of Eq. (3.15) still holds for this case, except that

$$\left\{\begin{array}{l} \left[\!\left[\underline{a}_n \bullet \nabla u \right]\!\right] \leftrightarrow -\left[\!\left[\underline{\underline{R}} \right]\!\right] \underline{u} \\ \dot{\widehat{\underline{a}_n \bullet \nabla w}} \leftrightarrow -\dot{\widehat{\underline{\underline{R}}}}\, \underline{u} \end{array}\right. \tag{3.17}$$

Correspondences similar to those of Eqs. (3.15) and (3.17) can be established in general; applications include the governing system of equations of linear elasticity and many other problems. We notice that Eqs. (3.15) and (3.17) imply a new formula for the *Steklov-Poincaré* operator (i.e., the

jump of the normal derivative) at the discrete level, which is different to standard interpretations that have been presented by many authors (compare, for example, with [9] pp 3, 46 and 47, or [11] pp 3 and 4). Indeed, our formula for the *Steklov-Poincaré* operator is as follows:

$$-\left[\!\left[\underline{\underline{R}}\right]\!\right] \equiv -\underline{\underline{aR}} \tag{3.18}$$

In particular, it does not contain the right-hand side of the equation to be solved; winning thereby, in theoretical consistency. In this respect, we notice that our formula is applicable to any vector (function) independently of whether it is solution of the problem under consideration or not.

IV. THE DISCONTINUOUS MULTIPLIERS-FREE MATRIX-FORMULATION

Define

$$\underline{\underline{\bar{A}}} \equiv \underline{\underline{aA}} \tag{4.1}$$

Then the "original problem" consists in: "Given $\underline{\bar{f}} \in \bar{D}(\bar{\Omega})$, find $\underline{\bar{u}} \in \bar{D}(\bar{\Omega})$ such that

$$\underline{\underline{\bar{A}}}\,\underline{\bar{u}} = \underline{\bar{f}} = \begin{pmatrix} \underline{\bar{f}}_{\Pi} \\ \underline{\bar{f}}_{\Delta} \end{pmatrix}, \tag{4.2}$$

Observe that the fact that $\underline{\bar{f}} \in \bar{D}(\bar{\Omega})$ implies that $\underline{\bar{f}}_{\Delta} \in \tilde{D}_2(\Delta) = \tilde{D}(\Delta) \cap \bar{D}(\bar{\Omega})$. Hence,

$$\underline{\bar{f}}_{\Delta 2} = \underline{\underline{a}}\,\underline{\bar{f}}_{\Delta} = \underline{\bar{f}}_{\Delta} \text{ and } \underline{\bar{f}}_{\Delta 1} = \underline{\underline{j}}\,\underline{\bar{f}}_{\Delta} = 0 \tag{4.3}$$

Recall

$$\underline{\underline{A}} = \underline{\underline{L}} + \underline{\underline{R}} \text{ and } \underline{\underline{aL}} = \underline{\underline{L}} \tag{4.4}$$

Therefore,

$$\underline{\underline{\bar{A}}} = \underline{\underline{aA}} = \underline{\underline{L}} + \underline{\underline{aR}} = \underline{\underline{L}} + \left[\!\left[\underline{\underline{R}}\right]\!\right] \tag{4.5}$$

Hence, $\underline{\bar{u}} \in \tilde{D}(\bar{\Omega})$ is the solution of the original problem, if and only if,

$$\left.\begin{array}{l} \underline{\underline{L}}\,\underline{\bar{u}} = \underline{\bar{f}}_{\Pi} \\ \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{\bar{u}} = \underline{\bar{f}}_{\Delta} \\ \underline{\underline{j}}\,\underline{\bar{u}} = 0 \end{array}\right\} \tag{4.6}$$

The following Lemma will be used in what follows.

Lemma 4.1. *Let $\underline{u} \in \tilde{D}(\bar{\Omega})$, then the following assertions are equivalent:*

1.

$$\underline{\underline{\hat{R}}}^{T}\,\underline{u} = 0 \tag{4.7}$$

2.

$$\underline{\underline{j}}\,\dot{\underline{\underline{\widehat{R}}}}^{T}\,\underline{u}=0 \tag{4.8}$$

3.

$$\underline{\underline{j}}\,\underline{u}=0 \tag{4.9}$$

Proof. Recall that $\underline{\underline{A}}_{\Delta\Delta}$ is positive definite and observe, for any $\underline{u}\in\tilde{D}(\bar{\Omega})$ one has

$$\underline{u}\bullet\underline{\underline{j}}\,\dot{\underline{\underline{\widehat{R}}}}^{T}\,\underline{u}=\underline{u}\bullet\underline{\underline{j}}\,\underline{\underline{R}}^{T}\,\underline{\underline{j}}\,\underline{u}=\underline{\underline{j}}\,\underline{u}\bullet\underline{\underline{A}}_{\Delta\Delta}\underline{\underline{j}}\,\underline{u}\geq 0 \tag{4.10}$$

and the equal sign holds if and only if $\underline{\underline{j}}\,\underline{u}=0$. Then, the equivalence between Eqs. (4.8) and (4.9) is clear. On the other hand, it is straight forward to see that Eq. (4.9) implies (4.7) while (4.7) implies (4.8). Then, the proof is complete. ∎

Using this Lemma, the conditions of Eq. (4.6) can be restated as: $\bar{\underline{u}}\in\tilde{D}(\bar{\Omega})$ is the solution of the original problem, if and only if,

$$\left.\begin{array}{l}\underline{\underline{L}}\,\bar{\underline{u}}=\bar{\underline{f}}_{\Pi}\\ \dot{\underline{\underline{\llbracket R\rrbracket}}}\,\bar{\underline{u}}=\bar{\underline{f}}_{\Delta}\\ \dot{\underline{\underline{\widehat{R}}}}^{T}\,\bar{\underline{u}}=0\end{array}\right\} \tag{4.11}$$

Theorem 4.2. *Let $\bar{\underline{u}}\in\tilde{D}(\bar{\Omega})$, then the equation*

$$\left(\underline{\underline{L}}-\underline{\underline{R}}^{T}\,\underline{\underline{j}}+\underline{\underline{a}}\,\underline{\underline{R}}\right)\bar{\underline{u}}\equiv\left(\underline{\underline{L}}-\dot{\underline{\underline{\widehat{R}}}}^{T}+\dot{\underline{\underline{\llbracket R\rrbracket}}}\right)\bar{\underline{u}}=\bar{\underline{f}} \tag{4.12}$$

is satisfied, if and only if, $\bar{\underline{u}}$ is the solution of the original problem.

Proof. That Eq. (4.12) is fulfilled when $\bar{\underline{u}}$ is solution of the original problem follows from Eq.(4.11). Conversely, assume Eq. (4.12) is satisfied, then apply $\underline{\underline{j}}$ to it to obtain

$$\underline{\underline{j}}\,\dot{\underline{\underline{\widehat{R}}}}^{T}\,\bar{\underline{u}}=0 \tag{4.13}$$

Therefore, $\underline{\underline{j}}\,\bar{\underline{u}}=0$ by the Lemma; i.e., $\bar{\underline{u}}\in\bar{D}(\bar{\Omega})$. Using this fact, Eq. (4.12) reduces to

$$(\underline{\underline{L}}+\underline{\underline{a}}\,\underline{\underline{R}})\bar{\underline{u}}=\bar{\underline{f}}=\bar{\underline{f}}_{\Pi}+\bar{\underline{f}}_{\Delta} \tag{4.14}$$

Or, applying Eq. (4.5),

$$\underline{\underline{\bar{A}}}\,\bar{\underline{u}}=\bar{\underline{f}} \tag{4.15}$$

This completes the required proof. ∎

To finish up with this section, we define the matrix $\underline{\underline{G}}$ by

$$\underline{\underline{G}} \equiv \underline{\underline{L}} - \overset{\bullet}{\underline{\underline{\widehat{R}}}}^T + \left[\left[\underline{\underline{R}}\right]\right] = \underline{\underline{L}} - \underline{\underline{R}}^T \underline{\underline{j}} + \underline{\underline{aR}} \tag{4.16}$$

Then, in view of Eq. (4.12), the original problem can be stated as "Find $\underline{\bar{u}} \in \tilde{D}(\bar{\Omega})$ such that

$$\underline{\underline{G}}\,\underline{\bar{u}} = \underline{\bar{f}}" \tag{4.17}$$

We notice that the matrix $\underline{\underline{G}}$ is symmetric, by virtue of Eq. (3.11). Generally it is nonpositive-definite; albeit, it is saddle as will be seen later. The result just obtained is relevant since we have succeeded in deriving a formulation of the original problem in discontinuous functions without resource to Lagrange multipliers: the "multipliers-free matrix-formulation" in discontinuous functions. As explained in the Introduction, the "unified multipliers-free theory of iterative substructuring DDM" that motivates the title of this article will be based on it.

V. RELATION WITH LAGRANGE MULTIPLIERS FORMULATIONS

The formulation with Lagrange multipliers can be written as (see, for example, [21])

$$\begin{pmatrix} \underline{\underline{A}}_{\Pi\Pi} & \underline{\underline{A}}_{\Pi\Delta} & 0 \\ \underline{\underline{A}}_{\Delta\Pi} & \underline{\underline{A}}_{\Delta\Delta} & \underline{\underline{B}}_{\Delta}^T \\ 0 & \underline{\underline{B}}_{\Delta} & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_{\Pi} \\ \underline{u}_{\Delta} \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_{\Pi} \\ \underline{f}_{\Delta} \\ 0 \end{pmatrix} \tag{5.1}$$

Or, with our notation, this is

$$(\underline{\underline{L}} + \underline{\underline{R}})\underline{u} + \underline{\underline{B}}_{\Delta}^T \underline{\lambda} = \underline{f}_{\Pi} + \underline{f}_{\Delta} \tag{5.2}$$

On the other hand, in the formulation without Lagrange multipliers we have:

$$\left(\underline{\underline{L}} - (\underline{\underline{R}})^T \underline{\underline{j}} + \underline{\underline{aR}}\right) \underline{\bar{u}} = \underline{f}_{\Pi} + \underline{f}_{\Delta} \tag{5.3}$$

Eqs. (5.2) and (5.3), together, imply

$$\underline{\underline{B}}_{\Delta}^T \underline{\lambda} = -\left(\underline{\underline{jR}} + (\underline{\underline{jR}})^T\right) \underline{\bar{u}} \tag{5.4}$$

When Eq. (5.4) is used in Eq.(5.2) it is seen that the "Lagrange multiplier formulation reduces" to the formulation without Lagrange multipliers.

VI. HARMONIC-VECTORS SPACES AND SUBSPACES

The inner product used in this section, unless otherwise explicitly stated, is understood to be the energy inner product; in particular, orthogonal complements will be taken with respect to it.

The "harmonic functions space" is defined to be

$$D \equiv \left\{ \underline{u} \in \tilde{D}(\bar{\Omega}) \;\middle|\; \underline{\underline{L}}\ \underline{u} = 0 \right\} \tag{6.1}$$

Members of D will be referred as "harmonic functions (or vectors)". An equivalent definition of this linear subspace of $\tilde{D}(\bar{\Omega})$ is that it is the orthogonal complement of $\tilde{D}(\Pi)$; i.e.,

$$D = \tilde{D}(\Pi)^{\perp} \tag{6.2}$$

Eq. (6.2) can be derived using the fact that the range of $\underline{\underline{L}}$ is contained in $\tilde{D}(\Pi)$, so that

$$\underline{w} \bullet \underline{\underline{L}}\underline{u} = \underline{w}_{\Pi} \bullet \underline{\underline{L}}\underline{u} = \underline{w}_{\Pi} \bullet \underline{\underline{A}}\underline{u}, \quad \forall \underline{w} \in \tilde{D}(\bar{\Omega}) \tag{6.3}$$

Furthermore, the transformation

$$Proj_D : \tilde{D}(\Delta) \to D \tag{6.4}$$

is one-to-one and covers D. This, because each one of the pairs $\{\tilde{D}(\Pi), \tilde{D}(\Delta)\}$ and $\{\tilde{D}(\Pi), D\}$ are linearly independent and

$$\tilde{D}(\Pi) + D = \tilde{D}(\bar{\Omega}) = \tilde{D}(\Pi) + \tilde{D}(\Delta) \tag{6.5}$$

The following subspaces of D will be used in the sequel:

$$\begin{aligned} &D_{11} \equiv \{\underline{u} \in D | \left[\!\left[\underline{\underline{R}}\right]\!\right]^T \underline{u} = 0\} \text{ and } D_{12} \equiv \left\{\underline{u} \in D \left| \overset{\bullet}{\underline{\underline{R}}}^T \underline{u} = 0 \right.\right\} \\ &D_{21} \equiv \left\{\underline{u} \in D \left| \overset{\bullet}{\underline{\underline{R}}}\underline{u} = 0 \right.\right\} \text{and} D_{22} \equiv \left\{\underline{u} \in D \left| \left[\!\left[\underline{\underline{R}}\right]\!\right] \underline{u} = 0 \right.\right\} \end{aligned} \tag{6.6}$$

We notice that

$$\begin{aligned} &D_{11} = \left\{\underline{u} \in D \left| \underline{\underline{a}}\underline{u}_{\Delta} = 0 \right.\right\} \text{ and } D_{12} = \left\{\underline{u} \in D \left| \underline{\underline{j}}\underline{u} = \underline{\underline{j}}\underline{u}_{\Delta} = 0 \right.\right\} \\ &D_{21} = \left\{\underline{u} \in D \left| \underline{\underline{j}}\underline{\underline{R}}\underline{u} = 0 \right.\right\} \text{ and } D_{22} = \left\{\underline{u} \in D \left| \underline{\underline{a}}\underline{\underline{R}}\underline{u} = 0 \right.\right\} \end{aligned} \tag{6.7}$$

That $\overset{\bullet}{\underline{\underline{R}}}^T \underline{u} = 0$ if and only if $\underline{\underline{j}}\underline{u} = 0$ was shown in Lemma 4.1. Similarly, it can be shown that $\left[\!\left[\underline{\underline{R}}\right]\!\right]^T \underline{u} = 0$ if and only if $\underline{\underline{a}}\underline{u}_{\Delta} = 0$. It can also be seen that

$$D_{11} \equiv Proj_D \tilde{D}_1(\Delta) \text{ and } D_{12} \equiv Proj_D \tilde{D}_2(\Delta) \tag{6.8}$$

Therefore

$$D = D_{11} \oplus D_{12} \tag{6.9}$$

The identity

$$\underline{\underline{A}} = \underline{\underline{L}} + \underline{\underline{R}} = \underline{\underline{L}} + \underline{\underline{a}}\underline{\underline{R}} + \underline{\underline{j}}\underline{\underline{R}} = \underline{\underline{L}}^T + \underline{\underline{R}}^T\underline{\underline{a}} + \underline{\underline{R}}^T\underline{\underline{j}} \tag{6.10}$$

will be used in what follows. Applying it, it is seen that

$$\begin{cases} \underline{w} \bullet \underline{\underline{A}}\underline{u} = 0, \forall \underline{u} \in D_{11} \text{ and } \underline{w} \in D_{21} \\ \underline{w} \bullet \underline{\underline{A}}\underline{u} = 0, \forall \underline{u} \in D_{12} \text{ and } \underline{w} \in D_{22} \end{cases} \tag{6.11}$$

Furthermore, it can also be seen that

$$\begin{cases} \underline{w} \bullet \underline{\underline{A}}\,\underline{u} = 0, \forall \underline{w} \in D_{11} \Rightarrow \underline{u} \in D_{21} \\ \underline{w} \bullet \underline{\underline{A}}\,\underline{u} = 0, \forall \underline{w} \in D_{12} \Rightarrow \underline{u} \in D_{22} \end{cases} \tag{6.12}$$

Therefore

$$D_{21} = D_{11}^{\perp} \text{ and } D_{22} = D_{12}^{\perp} \tag{6.13}$$

In [13], it was shown that Eq. (6.13) in the presence of Eq. (6.9), implies

$$D = D_{21} \oplus D_{22} \tag{6.14}$$

In summary, we have introduced two pairs of subspaces of D; namely, $\{D_{11}, D_{12}\}$ and $\{D_{21}, D_{22}\}$, which enjoy the following properties:

$$\begin{cases} D = D_{11} \oplus D_{12} \\ D = D_{21} \oplus D_{22} \\ D = D_{11} \oplus D_{21} \\ D = D_{12} \oplus D_{22} \\ D_{21} = D_{11}^{\perp} \text{ and } D_{22} = D_{12}^{\perp} \end{cases} \tag{6.15}$$

A geometrical interpretation of Eqs. (6.15) is supplied in Fig. 1.

Next we establish some properties of harmonic functions and, for this purpose, the functions $\underline{u}$ and $\underline{w}$ to be considered in what follows will be assumed to be in D. For such functions, Eq. (6.10) implies that when $\underline{u}, \underline{w} \in D$, one has:

$$\underline{w}\,\underline{\underline{A}}\,\underline{u} = \underline{w} \bullet \underline{\underline{a}}\,\underline{\underline{R}}\,\underline{u} + \underline{w} \bullet \underline{\underline{j}}\,\underline{\underline{R}}\,\underline{u} = \underline{w} \bullet \underline{\underline{R}}^T\,\underline{\underline{a}}\,\underline{u} + \underline{w} \bullet \underline{\underline{R}}^T\,\underline{\underline{j}}\,\underline{u} \tag{6.16}$$

Then

$$\underline{w}\,\underline{\underline{A}}\,\underline{u} = \underline{w} \bullet \underline{\underline{R}}^T\,\underline{\underline{a}}\,\underline{u}, \qquad \text{when } \underline{u} \in D_{12} \tag{6.17}$$

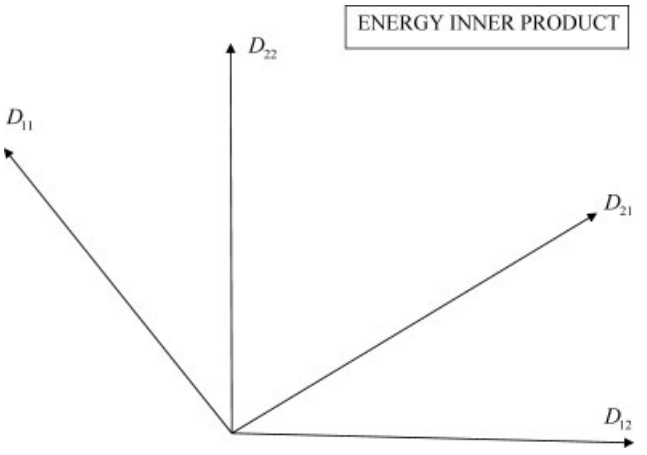

FIG. 1. Geometric summary.

and

$$\underline{w}\,\underline{\underline{A}}\,\underline{u} = \underline{w} \bullet \underline{\underline{j}}\,\underline{\underline{R}}\,\underline{u}, \quad \text{when } \underline{u} \in D_{22} \tag{6.18}$$

Using Eqs. (6.17) and (6.18) it can be seen that the bilinear form associated with $\underline{\underline{a}}\,\underline{\underline{R}}$ is symmetric and positive definite when restricted to functions in D_{12}, while the bilinear form associated with $\underline{\underline{j}}\,\underline{\underline{R}}$ is symmetric and positive definite when restricted to functions in D_{22}. Hence, the bilinear form associated with the matrix

$$\underline{\underline{G}} \equiv \underline{\underline{L}} - \underline{\underline{R}}^T\,\underline{\underline{j}} + \underline{\underline{a}}\,\underline{\underline{R}} \tag{6.19}$$

in terms of which the original problem was formulated in Eq. (4.17), is saddle, when it is restricted to harmonic functions; indeed, it is positive definite on D_{12} and it is negative definite on D_{22}.

VII. ONE-LEVEL METHODS, INCLUDING SCHUR COMPLEMENT, AND FETI

The purpose of this section is to transform the original problem into one that is formulated in terms of harmonic functions exclusively. To this end, first a general basic matrix equation is established, that will be used to derive one-level and two-level methods. In this section, it is applied to obtain the Schur-complement method, using a Dirichlet approach, and a one-level FETI method, using a Neumann approach. In the next section, the general basic matrix equation mentioned earlier is applied to derive two-level methods.

We start by defining the "dual-primal Schur complement matrix", $\underline{\underline{S}} : \tilde{D}(\Delta) \to \tilde{D}(\Delta)$ to be:

$$\underline{\underline{S}} \equiv \underline{\underline{A}}_{\Delta\Delta} - \underline{\underline{A}}_{\Delta\Pi}\,\underline{\underline{A}}_{\Pi\Pi}^{-1}\,\underline{\underline{A}}_{\Pi\Delta} \tag{7.1}$$

When $\underline{\underline{S}}$ is so defined, it is symmetric and positive definite; a property that will used in the sequel. Furthermore, when $\underline{\mathrm{v}} \in D$ one has

$$\underline{\underline{A}}\,\underline{\mathrm{v}} = \underline{\underline{R}}\,\underline{\mathrm{v}} = \begin{pmatrix} \underline{0} \\ \underline{\underline{S}}\,\underline{\mathrm{v}}_\Delta \end{pmatrix} = \underline{\underline{S}}\,\underline{\mathrm{v}}_\Delta \tag{7.2}$$

Let $\underline{\overline{u}} \in \overline{D}(\bar{\Omega})$ be solution of Eq. (4.12), and $\underline{\tilde{u}}_P \in \tilde{D}(\bar{\Omega})$ be any vector such that

$$\underline{\underline{L}}\,\underline{\tilde{u}}_P = \underline{\bar{f}}_\Pi \tag{7.3}$$

Then, we define

$$\underline{u} \equiv \underline{\bar{u}} - \underline{\tilde{u}}_P \tag{7.4}$$

and notice that $\underline{u} \in D$. In view of Eq. (4.12), it is seen that this vector satisfies

$$\left(\underline{\underline{a}}\,\underline{\underline{R}} - \underline{\underline{R}}^T\,\underline{\underline{j}}\right)\underline{u} = \underline{\bar{f}}_\Delta - \left(\underline{\underline{a}}\,\underline{\underline{R}} - \underline{\underline{R}}^T\,\underline{\underline{j}}\right)\underline{\tilde{u}}_P \tag{7.5}$$

Lemma 7.1. *Let $\underline{\mathrm{v}} \in D$, then, $\underline{\hat{u}} \equiv \underline{\tilde{u}}_P + \underline{\mathrm{v}}$ is solution of the original problem if and only if*

$$\underline{w} \bullet \left(\underline{\underline{a}}\,\underline{\underline{R}} - \underline{\underline{R}}^T\,\underline{\underline{j}}\right)\underline{\mathrm{v}} = \underline{w} \bullet \left(\underline{\bar{f}}_\Delta - \left(\underline{\underline{a}}\,\underline{\underline{R}} - \underline{\underline{R}}^T\,\underline{\underline{j}}\right)\underline{\tilde{u}}_P\right), \quad \forall \underline{w} \in D \tag{7.6}$$

Proof. Eq. (7.6) is a necessary condition in view of Eq. (7.5). Conversely, assume Eq. (7.6) is fulfilled, then

$$\underline{w} \bullet \left(\underline{\underline{a}}\,\underline{\underline{R}} - \underline{\underline{R}}^T\,\underline{\underline{j}}\right)\left(\underline{v} - \underline{u}\right) = 0, \quad \forall \underline{w} \in D = D_{12} \oplus D_{22} \tag{7.7}$$

Hence:

$$\begin{cases} \underline{w} \bullet \underline{\underline{a}}\,\underline{\underline{R}}(\underline{v} - \underline{u}) = \underline{w} \bullet (\underline{\underline{a}} + \underline{\underline{j}})\underline{\underline{R}}(\underline{v} - \underline{u}) = \underline{w} \bullet \underline{\underline{A}}(\underline{v} - \underline{u}) = 0, \quad \forall \underline{w} \in D_{12} \\ -\underline{w} \bullet \underline{\underline{R}}^T\,\underline{\underline{j}}(\underline{v} - \underline{u}) = -(\underline{v} - \underline{u}) \bullet \underline{\underline{j}}\,\underline{\underline{R}}\underline{w} = -(\underline{v} - \underline{u}) \bullet (\underline{\underline{a}} + \underline{\underline{j}})\underline{\underline{R}}\underline{w} = \\ -\underline{w} \bullet \underline{\underline{A}}(\underline{v} - \underline{u}) = 0, \quad \forall \underline{w} \in D_{22} \end{cases} \tag{7.8}$$

Eq. (7.8) implies that $(\underline{v} - \underline{u}) \in D_{12} \cap D_{22} = \{0\}$. That is $\underline{v} = \underline{u}$. Thus Eq. (7.6) is a sufficient condition. This completes the proof of the Lemma. ∎

Theorem 7.2. *Let* $\tilde{\underline{v}} \in \tilde{D}(\Omega)$ *be such that*

$$\tilde{\underline{v}}_\Pi = -\underline{\underline{A}}_{\Pi\Pi}^{-1}\underline{\underline{A}}_{\Pi\Delta}\tilde{\underline{v}}_\Delta \tag{7.9}$$

Then, $\underline{\tilde{u}} \equiv \underline{\tilde{u}}_P + \tilde{\underline{v}}$ *is the solution of the original problem if and only if*

$$\left(\underline{\underline{a}}\,\underline{\underline{S}} - \underline{\underline{S}}\,\underline{\underline{j}}\right)\tilde{\underline{v}}_\Delta = \bar{\underline{f}}_\Delta - \left(\underline{\underline{a}}\,\underline{\underline{R}} - \underline{\underline{S}}\,\underline{\underline{j}}\right)\tilde{\underline{u}}_P \tag{7.10}$$

Proof. Eq. (7.9) is a necessary and sufficient condition for $\tilde{\underline{v}} \in D$, and when $\tilde{\underline{v}} \in D$ then Eq. (7.6) is equivalent to

$$\underline{w}_\Delta \bullet \left(\underline{\underline{a}}\,\underline{\underline{S}} - \underline{\underline{S}}\,\underline{\underline{j}}\right)\tilde{\underline{v}}_\Delta = \underline{w}_\Delta \bullet \left(\bar{\underline{f}}_\Delta - \left(\underline{\underline{a}}\,\underline{\underline{R}} - \underline{\underline{S}}\,\underline{\underline{j}}\right)\tilde{\underline{u}}_P\right), \quad \forall \underline{w}_\Delta \in \tilde{D}(\Delta) \tag{7.11}$$

Finally, Eq. (7.11) is in turn equivalent to (7.10).

Eq. (7.10) is the basic matrix formulation without recourse to Lagrange multipliers mentioned at the beginning of this section, and we proceed to use it to derive one-level methods. In the development of iterative substructuring methods, the following two approaches have been extensively used.

The Dirichlet Approach: Assume, $\tilde{\underline{u}}_P \in \tilde{D}(\bar{\Omega})$ satisfies the equations

$$\left.\begin{array}{l} \underline{\underline{L}}\,\tilde{\underline{u}}_P = \bar{\underline{f}}_\Pi \\ \underline{\underline{j}}\,\tilde{\underline{u}}_P = 0 \end{array}\right\} \tag{7.12}$$

Then

$$\left.\begin{array}{l} \underline{\underline{L}}\,\underline{u} = 0 \\ \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{u} = \bar{\underline{f}}_\Delta - \left[\!\left[\underline{\underline{R}}\right]\!\right]\tilde{\underline{u}}_P \\ \underline{\underline{\hat{R}}}^T\,\underline{u} = 0 \end{array}\right\} \tag{7.13}$$

We notice that when $\tilde{\underline{u}}_P \in \tilde{D}(\bar{\Omega})$ is defined by Eq. (7.12), it is nonunique. However, if we choose $(\tilde{\underline{u}}_P)_\Delta = 0$, then $\tilde{\underline{u}}_P$ is the unique solution of a Dirichlet problem, and it is given by

$$\tilde{\underline{u}}_P = \begin{pmatrix} (\tilde{\underline{u}}_P)_\Pi \\ (\tilde{\underline{u}}_P)_\Delta \end{pmatrix} \text{ with } (\tilde{\underline{u}}_P)_\Pi \equiv \underline{\underline{A}}_{\Pi\Pi}^{-1}\bar{\underline{f}}_\Pi \text{ and } (\tilde{\underline{u}}_P)_\Delta \equiv 0 \tag{7.14}$$

In this case, Eq. (7.10) reduces to

$$\underline{\underline{aS}}\underline{u}_{\Delta} = \underline{\bar{f}}_{\Delta} - \underline{\underline{aR}}\tilde{\underline{u}}_P = \underline{\bar{f}}_{\Delta} - \underline{\underline{aA}}_{\Delta\Pi}\underline{\underline{A}}^{-1}_{\Pi\Pi}\underline{\bar{f}}_{\Pi} \tag{7.15}$$

Here, the matrix $\underline{\underline{aS}}$ is symmetric and positive definite on D_{12}.

We first notice that Eq. (7.15) corresponds to a standard formula [11], but now it applies to the dual-primal approach as well. So, by suitably choosing the dual nodes, the different dual-primal preconditioners are obtained. Second, we also notice that

$$\underline{w} \bullet \underline{\underline{Au}} = \underline{w} \bullet \underline{\underline{Su}}_{\Delta} = \underline{w} \bullet \left(\underline{\bar{f}}_{\Delta} - \underline{\underline{A}}_{\Delta\Pi}\underline{\underline{A}}^{-1}_{\Pi\Pi}\underline{\bar{f}}_{\Pi}\right), \quad \forall \underline{w} \in D_{12} \tag{7.16}$$

We notice that the matrix $\underline{\underline{a}}$ is not used when attention is restricted to continuous functions, as it is done in standard approaches.

Neumann Approach: Assume, $\tilde{\underline{u}}_P \in \tilde{D}(\bar{\Omega})$ satisfies the equations

$$\left.\begin{aligned} \underline{L}\tilde{\underline{u}}_P &= \underline{\bar{f}}_{\Pi} \\ \left[\!\left[\underline{\underline{R}}\right]\!\right]\tilde{\underline{u}}_P &= \underline{\bar{f}}_{\Delta} \end{aligned}\right\} \tag{7.17}$$

and define

$$u \equiv \bar{\underline{u}} - \tilde{\underline{u}}_P \tag{7.18}$$

Then

$$\left.\begin{aligned} \underline{Lu} &= 0 \\ \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{u} &= 0 \\ \underline{\underline{\hat{R}}}^{T}\,\underline{u} &= -\underline{\underline{\hat{R}}}^{T}\,\tilde{\underline{u}}_P \end{aligned}\right\} \tag{7.19}$$

Again, we notice that when $\tilde{\underline{u}}_P \in \tilde{D}(\bar{\Omega})$ is defined by Eq. (7.12) it is nonunique. However, if we impose the additional condition $\underline{\underline{\hat{R}}}\,\tilde{\underline{u}}_P = 0$, then $\tilde{\underline{u}}_P$ is the unique solution of a Neumann problem, and it is given by

$$\underline{u}_P = \underline{\underline{A}}^{-1}\underline{\bar{f}} \text{ with } \underline{\bar{f}} = \begin{pmatrix} \underline{\bar{f}}_{\Pi} \\ \underline{\bar{f}}_{\Delta} \end{pmatrix} \tag{7.20}$$

Eq. (7.10) reduces to

$$\underline{\underline{S}}\,\underline{\underline{j}}\,\underline{u}_{\Delta} = -\underline{\underline{S}}\,\underline{\underline{j}}\,\tilde{\underline{u}}_P \tag{7.21}$$

which is fulfilled if and only if

$$\underline{\underline{j}}\,\underline{u}_{\Delta} = -\underline{\underline{j}}\,\tilde{\underline{u}}_P \tag{7.22}$$

On the other hand, in this case the function $\underline{u} \in D$ is determined by the vector $\underline{\underline{\hat{R}}}\,\underline{u} = \underline{\underline{j}}\,\underline{\underline{S}}\underline{u}_{\Delta}$, since $\left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{u} = 0$ and $\underline{\underline{Au}} = \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{u} + \underline{\underline{\hat{R}}}\,\underline{u}$. So, we search for a value of $\underline{\underline{j}}\,\underline{\underline{S}}\underline{u}_{\Delta}$ with the property Eq. (7.21) is fulfilled. Let us call $\underline{\lambda} \equiv \underline{\underline{j}}\,\underline{\underline{S}}\underline{u}_{\Delta}$. Then

$$\underline{\underline{A}}\underline{u} = \underline{\lambda} \tag{7.23}$$

And Eq. (7.22) is equivalent to

$$\underline{\underline{j}}\,\underline{\underline{A}}^{-1}\underline{\lambda} = -\underline{\underline{j}}\,\underline{u}_P \tag{7.24}$$

since

$$\underline{u} = \underline{\underline{A}}^{-1}\underline{\lambda} \tag{7.25}$$

We notice that $\underline{\lambda} \in \tilde{D}_1(\Delta)$ necessarily since $\underline{\underline{a}}\,\underline{\lambda} = 0$. Therefore, our search for $\underline{\lambda}$ is on $\tilde{D}_1(\Delta)$ and the matrix $\underline{\underline{j}}\,\underline{\underline{A}}^{-1}$ is positive definite there. So, an iterative CGM approach is adequate to solve Eq. (7.24). Once $\underline{\lambda}$ has been obtained, $\underline{u}$ is given by Eq. (7.25).

Before we leave this section, a few auxiliary results that will be used in the sequel will be established. When the Dirichlet approach is used, the problem can be stated as:

"Find $\underline{u} \in D_{12} \subset D$ such that

$$\left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{u} = \underline{\bar{f}}_{\Delta} - \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{\tilde{u}}_P\text{"} \tag{7.26}$$

Theorem 7.3. *There exists a unique function $\underline{u}_{21} \in D$ that satisfies the conditions*

$$\left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{u}_{21} = \underline{\bar{f}}_{\Delta} - \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{\tilde{u}}_P \text{ and } \dot{\underline{\underline{R}}}\,\underline{u}_{21} = 0 \tag{7.27}$$

Such a function belongs to D_{21}. Furthermore, a vector $\underline{u} \in D_{12}$ is the solution of the Dirichlet approach if and only if, there exists a vector $\underline{u}_{22} \in D_{22}$ such that

$$\underline{u} = \underline{u}_{21} + \underline{u}_{22} \tag{7.28}$$

Proof. First, we show that there exists a unique function $\underline{u}_{21} \in D$ such that fulfills Eq. (7.27). Now, under the condition $\underline{u}_{21} \in D$ this latter equation is equivalent to

$$\underline{\underline{A}}\,\underline{u} = \left(\underline{\underline{L}} + \left[\!\left[\underline{\underline{R}}\right]\!\right] + \dot{\underline{\underline{R}}}\right)\underline{u} = \left(\underline{\underline{L}} + \left[\!\left[\underline{\underline{R}}\right]\!\right]\right)\underline{u} = \underline{\bar{f}}_{\Delta} - \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{\tilde{u}}_P \tag{7.29}$$

And Eq. (7.29) has a unique solution, since $\underline{\underline{A}}$ is positive definite. Now, when $\underline{u} \in D_{12}$ is the solution of the Dirichlet approach, define $\underline{u}_{22} \equiv \underline{u} - \underline{u}_{21}$. Then

$$\left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{u}_{22} = \left[\!\left[\underline{\underline{R}}\right]\!\right](\underline{u} - \underline{u}_{21}) = \underline{\bar{f}}_{\Delta} - \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{\tilde{u}}_P - (\underline{\bar{f}}_{\Delta} - \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{\tilde{u}}_P) = 0 \tag{7.30}$$

This implies $\underline{u}_{22} \in D_{22}$, and Eq. (7.28) is satisfied by virtue of the definition of $\underline{u}_{22}$. Conversely, assume $\underline{u} \in D_{12}$ and the conditions of Eq. (7.28) are satisfied, then

$$\left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{u} = \left[\!\left[\underline{\underline{R}}\right]\!\right](\underline{u}_{21} + \underline{u}_{22}) = \underline{\bar{f}}_{\Delta} - \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{\tilde{u}}_P \tag{7.31}$$

When the Neumann approach is used, the problem can be stated as follows:

"Find $\underline{u} \in D_{22} \subset D$ such that

$$\dot{\underline{\underline{\widehat{R}}}}^T\,\underline{u} = -\dot{\underline{\underline{\widehat{R}}}}^T\,\underline{\tilde{u}}_P \tag{7.32}$$

or, equivalently

$$\underline{\underline{j}}\,\underline{u} = -\underline{\underline{j}}\,\underline{\tilde{u}}_P \text{ "} \tag{7.33}$$

■

Theorem 7.4. *There exists a unique function* $\underline{u}_{11} \in D$ *that satisfies the conditions*

$$\underline{\underline{j}}(\underline{u}_{11} + \underline{\tilde{u}}_P) = 0 \text{ and } \underline{\underline{a}}(\underline{u}_{11})_\Delta = 0 \tag{7.34}$$

Furthermore, $\underline{u}_{11} \in D_{11}$ *and a vector* $\underline{u} \in D_{22}$ *is the solution of the Neumann approach if and only if, there exists a function* $\underline{u}_{12} \in D_{12}$ *such that:*

$$\underline{u} = \underline{u}_{11} + \underline{u}_{12} \tag{7.35}$$

Proof. First, we show that there exists a unique function $\underline{u}_{11} \in D$ that fulfills Eq. (7.34). To this end observe that this latter equation implies that

$$(\underline{u}_{11})_\Delta = -\underline{\underline{j}}\,\underline{\tilde{u}}_P \tag{7.36}$$

Furthermore, the condition $\underline{u}_{11} \in D$ is equivalent to

$$\underline{\underline{L}}\,\underline{u} = 0 \tag{7.37}$$

Using the definition of $\underline{\underline{L}}$, Eq. (3.4), we have

$$\underline{\underline{A}}_{\Pi\Pi}(\underline{u}_{11})_\Pi = -\underline{\underline{A}}_{\Pi\Delta}(\underline{u}_{11})_\Delta \tag{7.38}$$

This determines $\underline{u}_{11} \in D$ uniquely, since $\underline{\underline{A}}_{\Pi\Pi}$ is positive definite. Now, assume $\underline{u} \in D_{22}$ is the solution of the Neumann approach, and then define $\underline{u}_{12} \equiv \underline{u} - \underline{u}_{11}$. Hence,

$$\underline{\underline{j}}\,\underline{u}_{12} = \underline{\underline{j}}(\underline{u} - \underline{u}_{11}) = \underline{\underline{j}}(\underline{u} + \underline{\tilde{u}}_P) = 0 \tag{7.39}$$

This implies $\underline{u}_{12} \in D_{12}$, and it can be verified that Eq. (7.35) is satisfied. Conversely, assume $\underline{u} \in D_{22}$ and the conditions of Eq. (7.35) are satisfied, then

$$\underline{\underline{j}}\,\underline{u} = \underline{\underline{j}}(\underline{u}_{11} + \underline{u}_{12}) = \underline{\underline{j}}\,\underline{u}_{11} = -\underline{\underline{j}}\,\underline{\tilde{u}}_P \tag{7.40}$$

■

VIII. TWO-LEVEL METHODS AND THE 'ROUND-TRIP' ALGORITHM

In this section, we briefly revise a general framework that was first introduced in [13] and two-level methods are derived from it.

In Section VII, Theorems 7.3 and 7.4, we were led to consider the following two abstract problems.

Problem 1. In this problem, $\underline{u}_{21} \in D_{21}$ is the datum: "Given $\underline{u}_{21} \in D_{21}$, find $\underline{u} \in D_{12}$ such that

$$\underline{u} = \underline{u}_{21} + \underline{u}_{22} \text{ for some } \underline{u}_{22} \in D_{22}\text{"} \tag{8.1}$$

Problem 2. In this problem, $\underline{u}_{11} \in D_{11}$ is the datum: "Given $\underline{u}_{11} \in D_{11}$, find $\underline{u} \in D_{22}$ such that

$$\underline{u} = \underline{u}_{11} + \underline{u}_{12} \text{ for some } \underline{u}_{12} \in D_{12}" \tag{8.2}$$

At the end of this section a Theorem is given, which yields formulations of these problems that are convenient in many respects. They are based on some linear transformations that we introduce next. Eq. (6.15) implies that every function $\underline{u} \in D$ can be written in a unique manner as

$$\underline{u} = \underline{u}_{11} + \underline{u}_{12} = \underline{u}_{21} + \underline{u}_{22}, \text{ with } \underline{u}_{\alpha\beta} \in D_{\alpha\beta}; \alpha, \beta = 1, 2 \tag{8.3}$$

So, we define for each $\alpha, \beta = 1, 2$, the matrices $\underline{\underline{\sigma}}_{\alpha\beta} : D \to D_{\alpha\beta}$, which satisfy for each $\underline{u} \in D$, the condition

$$\underline{\underline{\sigma}}_{\alpha\beta}\underline{u} \equiv \underline{u}_{\alpha\beta} \tag{8.4}$$

For greater clarity, we recall that the energy inner product will be used in what follows exclusively, as we have been doing.

Lemma 8.1. *We have*

$$\left.\begin{array}{c}(\underline{w}, \underline{\underline{\sigma}}_{12}\underline{\underline{\sigma}}_{21}\underline{u}) = (\underline{w}, \underline{u}) + (\underline{\underline{\sigma}}_{22}\underline{w}, \underline{\underline{\sigma}}_{22}\underline{u}) = (\underline{w}, \underline{u}) - (\underline{w}, \underline{\underline{\sigma}}_{12}\underline{\underline{\sigma}}_{22}\underline{u}) \\ \left(\underline{w}, \underline{\underline{\sigma}}_{12}\underline{\underline{\sigma}}_{22}\underline{u}\right) = -\left(\underline{\underline{\sigma}}_{22}\underline{w}, \underline{\underline{\sigma}}_{22}\underline{u}\right)\end{array}\right\}, \quad \forall \underline{u}, \underline{w} \in D_{12} \tag{8.5}$$

Together with

$$\left.\begin{array}{c}(\underline{w}, \underline{\underline{\sigma}}_{22}\underline{\underline{\sigma}}_{11}\underline{u}) = (\underline{w}, \underline{u}) + (\underline{\underline{\sigma}}_{12}\underline{w}, \underline{\underline{\sigma}}_{12}\underline{u}) = (\underline{w}, \underline{u}) - (\underline{w}, \underline{\underline{\sigma}}_{22}\underline{\underline{\sigma}}_{12}\underline{u}) \\ (\underline{w}, \underline{\underline{\sigma}}_{22}\underline{\underline{\sigma}}_{12}\underline{u}) = -(\underline{\underline{\sigma}}_{12}\underline{w}, \underline{\underline{\sigma}}_{12}\underline{u})\end{array}\right\} \quad \forall \underline{u}, \underline{w} \in D_{22} \tag{8.6}$$

Proof. For $\underline{u} \in D_{12}$, we observe that

$$\underline{\underline{\sigma}}_{21}\underline{u} = \underline{u} - \underline{\underline{\sigma}}_{22}\underline{u} \text{ and } \underline{u} \perp \underline{\underline{\sigma}}_{22}\underline{u} \tag{8.7}$$

Therefore,

$$\left(\underline{\underline{\sigma}}_{21}\underline{w}, \underline{\underline{\sigma}}_{21}\underline{u}\right) = \left(\underline{w} - \underline{\underline{\sigma}}_{22}\underline{w}, \underline{u} - \underline{\underline{\sigma}}_{22}\underline{u}\right) = \left(\underline{w}, \underline{u}\right) + \left(\underline{\underline{\sigma}}_{22}\underline{w}, \underline{\underline{\sigma}}_{22}\underline{u}\right) \tag{8.8}$$

Furthermore,

$$\left(\underline{\underline{\sigma}}_{22}\underline{w}, \underline{\underline{\sigma}}_{22}\underline{u}\right) = \left(\underline{\underline{\sigma}}_{22}\underline{w}, \underline{\underline{\sigma}}_{11}\underline{\underline{\sigma}}_{22}\underline{u}\right) = \left(\underline{w}, \underline{\underline{\sigma}}_{11}\underline{\underline{\sigma}}_{22}\underline{u}\right) = -\left(\underline{w}, \underline{\underline{\sigma}}_{12}\underline{\underline{\sigma}}_{22}\underline{u}\right) \tag{8.9}$$

Hence, Eq. (8.5) is clear. As for Eq. (8.6), it can be derived in a similar fashion. ∎

Corollary 8.2.

1. *The mapping $\underline{\underline{\sigma}}_{12}\underline{\underline{\sigma}}_{21} : D_{12} \to D_{12}$ is symmetric and positive definite on D_{12}. Furthermore*

$$\left(\underline{u}, \underline{\underline{\sigma}}_{12}\underline{\underline{\sigma}}_{21}\underline{u}\right) = \left(\underline{u}, \underline{u}\right) + \left(\underline{\underline{\sigma}}_{22}\underline{u}, \underline{\underline{\sigma}}_{22}\underline{u}\right) \geq \left(\underline{u}, \underline{u}\right), \forall \underline{u} \in D_{12} \tag{8.10}$$

2. *The mapping* $\underline{\underline{\sigma}}_{22}\underline{\underline{\sigma}}_{12} : D_{22} \to D_{22}$ *is symmetric and positive definite on* D_{22}. *Furthermore*

$$(\underline{u}, \underline{\underline{\sigma}}_{22}\underline{\underline{\sigma}}_{11}\underline{u}) = (\underline{u}, \underline{u}) + (\underline{\underline{\sigma}}_{12}\underline{u}, \underline{\underline{\sigma}}_{12}\underline{u}) \geq (\underline{u}, \underline{u}), \forall \underline{u} \in D_{22} \tag{8.11}$$

Proof. These results are straightforward applications of Lemma 8.1. ∎

Definition 8.3. *We define the operators* $T_D : D_{12} \to D_{12}$ *and* $T_N : D_{22} \to D_{22}$ *by*

$$T_D\underline{u} \equiv -\underline{\underline{\sigma}}_{12}\underline{\underline{\sigma}}_{22}\underline{u} \text{ and } T_N\underline{u} \equiv -\underline{\underline{\sigma}}_{22}\underline{\underline{\sigma}}_{12}\underline{u} \tag{8.12}$$

Remark 8.4. We notice that, by virtue of Eqs. (8.5) and (8.6), both $T_D : D_{12} \to D_{12}$ and $T_N : D_{22} \to D_{22}$ are symmetric and nonnegative. Furthermore:

$$\begin{cases} \underline{\underline{\sigma}}_{12}\underline{\underline{\sigma}}_{21}\underline{u} = (\mathrm{I} + T_D)\underline{u}, \forall \underline{u} \in D_{12} \\ \underline{\underline{\sigma}}_{22}\underline{\underline{\sigma}}_{11}\underline{u} = (\mathrm{I} + T_N)\underline{u}, \forall \underline{u} \in D_{22} \end{cases} \tag{8.13}$$

Theorem 8.5. *Formulation of Problems 1 and 2*

A. *Problem 1, possesses a unique solution, which satisfies*:

$$(I + T_D)\underline{u} = \underline{\underline{\sigma}}_{12}\underline{u}_{21} \tag{8.14}$$

B. *Problem 2, possesses a unique solution, which satisfies*:

$$(I + T_N)\underline{u} = \underline{\underline{\sigma}}_{22}\underline{u}_{11} \tag{8.15}$$

C. *In view of Remark 8.4, Eqs.* (8.14) *and* (8.15) *can also be written as*

$$\underline{\underline{\sigma}}_{12}\underline{\underline{\sigma}}_{21}\underline{u} = \underline{\underline{\sigma}}_{12}\underline{u}_{21} \text{ and } \underline{\underline{\sigma}}_{22}\underline{\underline{\sigma}}_{11}\underline{u} = \underline{\underline{\sigma}}_{22}\underline{u}_{11} \tag{8.16}$$

respectively.

Proof. Eq. (8.14) possesses a unique solution, $\bar{\underline{u}} \in D_{12}$, because $\underline{\underline{\sigma}}_{12}\underline{\underline{\sigma}}_{21}$ is positive definite on D_{12} and $\underline{\underline{\sigma}}_{12}\underline{u}_{21} \in D_{12}$. Then

$$\underline{\underline{\sigma}}_{12}(\underline{\underline{\sigma}}_{21}\bar{\underline{u}} - \underline{u}_{21}) = 0 \tag{8.17}$$

This implies that $(\underline{\underline{\sigma}}_{21}\bar{\underline{u}} - \underline{u}_{21}) \in D_{11} \cap D_{21} = \{0\}$; i.e.,

$$\underline{\underline{\sigma}}_{21}\bar{\underline{u}} = \underline{u}_{21} \tag{8.18}$$

Hence

$$\bar{\underline{u}} = \underline{u}_{21} + (\bar{\underline{u}})_{22} \tag{8.19}$$

is a solution of Problem 1. This proves that Eq. (8.14) is a sufficient condition and thereby existence of a solution for this problem. Assume now $\underline{u}_{21} \in D_{21}$, $\underline{u}_{22} \in D_{22}$, $\underline{u} \in D_{12}$ and

$$\underline{u} = \underline{u}_{21} + \underline{u}_{22} \tag{8.20}$$

Then, applying $\underline{\underline{\sigma}}_{21}$ one gets

$$\underline{\underline{\sigma}}_{21}\underline{u} = \underline{u}_{21} \tag{8.21}$$

Application of $\underline{\underline{\sigma}}_{12}$ to this equation yields Eq. (8.14). Therefore, that equation is a necessary condition. Thereby, this proves that the solution of Problem 1 is unique. The proof of Part *A* is now complete and the proof of Part *B* is similar. ∎

The algorithms derived from Theorem 8.5 will be referred to as "round-trip algorithms". We notice that the application of $I + T_D = \underline{\underline{\sigma}}_{12}\underline{\underline{\sigma}}_{21}$ consists of the solution of a Dirichlet problem followed by a Neumann one, whereas the application of $\mathrm{I} + T_N = \underline{\underline{\sigma}}_{22}\underline{\underline{\sigma}}_{1,1}$ consists of the solution of a Neumann problem followed by a Dirichlet one. So, the "round-trip algorithms" include methods that are similar, probably equivalent, to the well-known Neumann-Neumann and preconditioned FETI methods of the literature [9–11]. We observe, furthermore, that the conjugate gradient method can be applied to the "round-trip algorithms" since for them the system-matrix is positive definite.

IX. TWO-LEVEL DUAL-PRIMAL MATRIX EQUATIONS

Application of the round-trip algorithm requires computing the components $\underline{\mathrm{v}}_{\gamma\alpha} \in D_{\gamma\alpha}$, of any given a vector $\underline{\mathrm{v}} \in D$, which have the property that

$$\underline{\mathrm{v}} = \sum_{\alpha=1}^{2} \underline{\mathrm{v}}_{\gamma\alpha}, \gamma = 1, 2 \tag{9.1}$$

In this section, the bases for carrying out such computations are established.

According to Eq. (2.28), when $\underline{\mathrm{v}} \in D$, we can write

$$\underline{\mathrm{v}} = \underline{\mathrm{v}}_{\Delta 1} + \underline{v}_{\Delta 2} + \underline{\mathrm{v}}_{\Pi}, \quad \text{where } \underline{\mathrm{v}}_{\Delta 1} \in \tilde{D}_1(\Delta), \underline{\mathrm{v}}_{\Delta 2} \in \tilde{D}_2(\Delta) \text{ and } \underline{\mathrm{v}}_{\Pi} \in \tilde{D}(\Pi) \tag{9.2}$$

Furthermore, $\underline{\mathrm{v}}_{\Delta 1} = (\underline{\mathrm{v}}_{11})_{\Delta 1}$ and $\underline{\mathrm{v}}_{\Delta 2} = (\underline{\mathrm{v}}_{12})_{\Delta 2}$ because

$$\begin{aligned} \underline{\mathrm{v}}_{11} &= (\underline{\mathrm{v}}_{11})_{\Delta 1} + (\underline{\mathrm{v}}_{11})_{\Pi}, \quad \text{where } (\underline{\mathrm{v}}_{11})_{\Delta 1} \in \tilde{D}_1(\Delta) \text{ and } (\underline{\mathrm{v}}_{11})_{\Pi} \in \tilde{D}(\Pi) \\ \underline{\mathrm{v}}_{12} &= (\underline{\mathrm{v}}_{12})_{\Delta 2} + (\underline{\mathrm{v}}_{12})_{\Pi}, \quad \text{where } (\underline{\mathrm{v}}_{12})_{\Delta 2} \in \tilde{D}_2(\Delta) \text{ and } (\underline{\mathrm{v}}_{12})_{\Pi} \in \tilde{D}(\Pi) \end{aligned} \tag{9.3}$$

Then the equations $\underline{\underline{L}}\underline{\mathrm{v}}_{11} = \underline{\underline{L}}\underline{\mathrm{v}}_{12} = 0$ together with Eqs. (9.2) and (9.3) imply

$$\begin{cases} ((\underline{\mathrm{v}}_{11})_{\Pi}, \tilde{w}) = -(\underline{\mathrm{v}}_{\Delta 1}, \tilde{w}), \forall \tilde{w} \in \tilde{D}(\Pi) \\ ((\underline{\mathrm{v}}_{12})_{\Pi}, \tilde{w}) = -(\underline{\mathrm{v}}_{\Delta 2}, \tilde{w}), \forall \tilde{w} \in \tilde{D}(\Pi) \end{cases} \tag{9.4}$$

Eqs. (9.4) yield

$$\left.\begin{aligned} \underline{\underline{A}}_{\Pi\Pi}(\underline{\mathrm{v}}_{11})_{\Pi} &= -\underline{\underline{A}}_{\Pi\Delta}\underline{\underline{j}}\underline{\mathrm{v}}_{\Delta} \\ \underline{\underline{A}}_{\Pi\Pi}(\underline{\mathrm{v}}_{12})_{\Pi} &= -\underline{\underline{A}}_{\Pi\Delta}\underline{\underline{a}}\underline{\mathrm{v}}_{\Delta} \end{aligned}\right\} \tag{9.5}$$

Using the facts that every $\underline{w} \in D_{11}$ is

$$\underline{w} = \underline{w}_{\Delta 1} + \underline{w}_{\Pi}, \quad \text{where } \underline{w}_{\Delta 1} \in \tilde{D}_1(\Delta) \text{ and } \underline{w}_{\Pi} \in \tilde{D}(\Pi) \tag{9.6}$$

and that every $\underline{w} \in D_{12}$ is

$$\underline{w} = \underline{w}_{\Delta 2} + \underline{w}_{\Pi}, \quad \text{where } \underline{w}_{\Delta 2} \in \tilde{D}_2(\Delta) \text{ and } \underline{w}_{\Pi} \in \tilde{D}(\Pi) \tag{9.7}$$

it is seen that

$$\begin{aligned}(\underline{\mathrm{v}}_{21}, \underline{\tilde{w}}) &= 0, \quad \forall \underline{\tilde{w}} \in \tilde{D}(\Pi) \\ (\underline{\mathrm{v}}_{21}, \underline{\tilde{w}}) &= (\underline{\mathrm{v}}, \underline{\tilde{w}}), \quad \forall \underline{\tilde{w}} \in \tilde{D}_2(\Delta) \\ (\underline{\mathrm{v}}_{21}, \underline{\tilde{w}}) &= 0, \quad \forall \underline{\tilde{w}} \in \tilde{D}_1(\Delta)\end{aligned} \tag{9.8}$$

together with

$$\begin{aligned}(\underline{\mathrm{v}}_{22}, \underline{\tilde{w}}) &= 0, \quad \forall \underline{\tilde{w}} \in \tilde{D}(\Pi) \\ (\underline{\mathrm{v}}_{22}, \underline{\tilde{w}}) &= 0, \quad \forall \underline{\tilde{w}} \in \tilde{D}_2(\Delta) \\ (\underline{\mathrm{v}}_{22}, \underline{\tilde{w}}) &= (\underline{\mathrm{v}}, \underline{\tilde{w}}), \quad \forall \underline{\tilde{w}} \in \tilde{D}_1(\Delta)\end{aligned} \tag{9.9}$$

Now, let $\underline{\tilde{w}} \in \tilde{D}(\Delta)$, then

$$\underline{\tilde{w}} = \underline{\underline{j}}\,\underline{\tilde{w}} + \underline{\underline{a}}\,\underline{\tilde{w}} \text{ where } \underline{\underline{j}}\,\underline{\tilde{w}} \in \tilde{D}_1(\Delta) \text{ and } \underline{\underline{a}}\,\underline{\tilde{w}} \in \tilde{D}_2(\Delta) \tag{9.10}$$

Thus, applying Eqs. (9.8) and (9.9) we get

$$(\underline{\mathrm{v}}_{21}, \underline{\tilde{w}}) = (\underline{\mathrm{v}}, \underline{\underline{a}}\,\underline{\tilde{w}}_{\Delta}), \forall \underline{\tilde{w}} \in \tilde{D}(\bar{\Omega}) \tag{9.11}$$

and

$$(\underline{\mathrm{v}}_{22}, \underline{\tilde{w}}) = (\underline{\mathrm{v}}, \underline{\underline{j}}\,\underline{\tilde{w}}_{\Delta}), \quad \forall \underline{\tilde{w}} \in \tilde{D}(\bar{\Omega}), \tag{9.12}$$

respectively. Eq. (9.11) can be written as

$$\underline{\tilde{w}} \bullet \underline{\underline{A}}\,\underline{\mathrm{v}}_{21} = \underline{\tilde{w}}_{\Delta} \bullet \underline{\underline{\underline{a}}}\,\underline{\underline{A}}\,\underline{\mathrm{v}} = \underline{\tilde{w}}_{\Delta} \bullet \underline{\underline{a}}(\underline{\underline{L}} + \underline{\underline{R}})\underline{\mathrm{v}} = \underline{\tilde{w}}_{\Delta} \bullet \underline{\underline{\underline{a}}}\,\underline{\underline{R}}\,\underline{\mathrm{v}}, \quad \forall \underline{w} \in \tilde{D}(\bar{\Omega}) \tag{9.13}$$

Furthermore,

$$\underline{\tilde{w}}_{\Pi} \bullet \underline{\underline{\underline{a}}}\,\underline{\underline{R}}\,\underline{\mathrm{v}} = (\underline{\underline{a}}\,\underline{\tilde{w}}_{\Pi}) \bullet \underline{\underline{R}}\,\underline{\mathrm{v}} = \underline{\tilde{w}}_{\Pi} \bullet \underline{\underline{R}}\,\underline{\mathrm{v}} = 0, \quad \forall \underline{w}_{\Pi} \in \tilde{D}(\Pi) \tag{9.14}$$

Hence, Eq. (9.13) can be written as

$$\underline{\tilde{w}} \bullet \underline{\underline{A}}\,\underline{\mathrm{v}}_{21} = \underline{\tilde{w}} \bullet \underline{\underline{\underline{a}}}\,\underline{\underline{R}}\,\underline{\mathrm{v}}, \quad \forall \underline{w} \in \tilde{D}(\bar{\Omega}) \tag{9.15}$$

This latter equation is equivalent to

$$\underline{\underline{A}}\,\underline{\mathrm{v}}_{21} = \underline{\underline{\underline{a}}}\,\underline{\underline{R}}\,\underline{\mathrm{v}} \tag{9.16}$$

Similarly, Eq. (9.12) is equivalent to

$$\underline{\underline{A}}\,\underline{\mathrm{v}}_{22} = \underline{\underline{j}}\,\underline{\underline{R}}\,\underline{\mathrm{v}} \tag{9.17}$$

Summarizing, let any function $\underline{\mathrm{v}} \in D$ be given by Eq. (9.2), then its components $\underline{\mathrm{v}}_{\gamma\alpha} \in D_{\gamma\alpha}$ that satisfy Eq. (9.1), can be obtained applying the following matrix equations

$$\left(\underline{\mathrm{v}}_{11}\right)_{\Pi} = -\left(\underline{\underline{A}}_{\Pi\Pi}\right)^{-1} \underline{\underline{A}}_{\Pi\Delta}\,\underline{\underline{j}}\,\underline{\mathrm{v}}_{\Delta} \text{ together with } \left(\underline{\mathrm{v}}_{12}\right)_{\Pi} = -\left(\underline{\underline{A}}_{\Pi\Pi}\right)^{-1} \underline{\underline{A}}_{\Pi\Delta}\,\underline{\underline{a}}\,\underline{\mathrm{v}}_{\Delta} \tag{9.18}$$

and

$$\underline{\mathrm{v}}_{21} = \underline{\underline{A}}^{-1}\underline{\underline{a}}\underline{\underline{R}}\underline{\mathrm{v}} \text{ together with } \underline{\mathrm{v}}_{22} = \underline{\underline{A}}^{-1}\underline{\underline{j}}\underline{\underline{R}}\underline{v} \tag{9.19}$$

Application of the round-trip algorithms require also the evaluation of the vectors $\underline{u}_{21} \in D_{21}$ and $\underline{u}_{11} \in D_{11}$, that were defined in Theorems 7.2 and 7.3, for the first approach and for the second approach, respectively. According to those theorems, such functions are given by

$$\underline{u}_{21} = \underline{\underline{A}}^{-1}\left(\underline{\bar{f}}_{\Delta} - \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{\tilde{u}}_P\right) \text{ and } \left(u_{11}\right)_{\Pi} = A^{-1}_{\Pi\Pi}A_{\Pi\Delta}\underline{\underline{j}}\tilde{u}_P \tag{9.20}$$

X. THE CASE WHEN ALL PRIMAL NODES ARE INTERIOR

This corresponds to case when $\{\Omega_1, \ldots, \Omega_E\}$ is a partition of $\bar{\Omega}$; i.e.,

$$\bar{\Omega} = \bigcup_{\alpha=1}^{E} \bar{\Omega}_\alpha \text{ and } \bar{\Omega}_\alpha \cap \bar{\Omega}_\beta = \varphi \quad \text{when } \alpha \neq \beta \tag{10.1}$$

We define

$$\Gamma_\alpha \equiv \Delta \cap \bar{\Omega}_\alpha, \mathrm{I}_\alpha \equiv \Pi \cap \bar{\Omega}_\alpha \text{ together with } \Gamma \equiv \bigcup_{\alpha=1}^{E} \Gamma_\alpha, \mathrm{I} \equiv \bigcup_{\alpha=1}^{E} \mathrm{I}_\alpha \tag{10.2}$$

Then

$$\bar{\Omega} = \Gamma \cup \mathrm{I} \quad \text{while } \Delta = \Gamma \text{ and } \Pi = \mathrm{I} \tag{10.3}$$

Furthermore, for each $\alpha = 1, \ldots, E$, we define the matrix

$$\underline{\underline{A}}^{\alpha} : D(\bar{\Omega}_\alpha) \to D(\bar{\Omega}_\alpha) \tag{10.4}$$

by the condition that the bilinear form associated to it is the restriction to $D(\bar{\Omega}_\alpha) \times D(\bar{\Omega}_\alpha)$ of the bilinear form associated with $\underline{\underline{A}}^{\alpha} : D(\bar{\Omega}) \to D(\bar{\Omega})$; more precisely:

$$\underline{w} \bullet \underline{\underline{A}}^{\alpha}\underline{\mathrm{v}} = \underline{w}^{\alpha} \bullet \underline{\underline{A}}\underline{\mathrm{v}}^{\alpha}, \quad \forall \underline{w}, \underline{\mathrm{v}} \in D(\bar{\Omega}_\alpha) \tag{10.5}$$

Here,

$$\underline{\mathrm{v}} = \sum_{\alpha=1}^{E} \underline{\mathrm{v}}^{\alpha} \text{ and } \underline{w} = \sum_{\alpha=1}^{E} \underline{w}^{\alpha}, \quad \text{with } \underline{\mathrm{v}}^{\alpha}, \underline{w}^{\alpha} \in D(\bar{\Omega}^{\alpha}) \tag{10.6}$$

Then, we write

$$\underline{\underline{A}}^{\alpha} \equiv \begin{pmatrix} \underline{\underline{A}}^{\alpha}_{\mathrm{II}} & \underline{\underline{A}}^{\alpha}_{\mathrm{I}\Gamma} \\ \underline{\underline{A}}^{\alpha}_{\Gamma\mathrm{I}} & \underline{\underline{A}}^{\alpha}_{\Gamma\Gamma} \end{pmatrix} \tag{10.7}$$

Furthermore, for the case discussed in this section one has

$$\underline{\underline{A}} \equiv \sum_{\alpha=1}^{E} \underline{\underline{A}}^{\alpha} \tag{10.8}$$

Therefore, we have

$$\begin{cases} \underline{\underline{A}}_{\Pi\Pi} \equiv \sum_{\alpha=1}^{E} \underline{\underline{A}}_{\Pi\Pi}^{\alpha}, \underline{\underline{A}}_{\Pi\Gamma} \equiv \sum_{\alpha=1}^{E} \underline{\underline{A}}_{\Pi\Gamma}^{\alpha} \\ \underline{\underline{A}}_{\Gamma\mathrm{I}} \equiv \sum_{\alpha=1}^{E} \underline{\underline{A}}_{\Gamma\mathrm{I}}^{\alpha}, \underline{\underline{A}}_{\Gamma\Gamma} \equiv \sum_{\alpha=1}^{E} \underline{\underline{A}}_{\Gamma\Gamma}^{\alpha} \end{cases} \tag{10.9}$$

We notice the following important properties

$$\left(\underline{\underline{A}}_{\Pi\Pi}\right)^{-1} = \sum_{\alpha=1}^{E} \left(\underline{\underline{A}}_{\Pi\Pi}^{\alpha}\right)^{-1} \text{ and } \left(\underline{\underline{A}}\right)^{-1} \equiv \sum_{\alpha=1}^{E} \left(\underline{\underline{A}}^{\alpha}\right)^{-1} \tag{10.10}$$

This implies that computing the inverses of the matrices $\underline{\underline{A}}_{\Pi\Pi}$ and $\underline{\underline{A}}$ requires computing local inverses exclusively. ∎

Eqs. (9.18) and (9.19) now become

$$(\underline{\mathrm{v}}_{11})_{\mathrm{I}} = -\sum_{\alpha=1}^{E} \left(\underline{\underline{A}}_{\Pi\Pi}^{\alpha}\right)^{-1} \underline{\underline{A}}_{\mathrm{I}\Gamma} \underline{\underline{j}} \underline{\mathrm{v}}_{\Gamma} \text{ together with } (\underline{\mathrm{v}}_{12})_{\mathrm{I}} = -\sum_{\alpha=1}^{E} \left(\underline{\underline{A}}_{\Pi\Pi}^{\alpha}\right)^{-1} \underline{\underline{A}}_{\mathrm{I}\Gamma} \underline{\underline{a}} \underline{\mathrm{v}}_{\Gamma} \tag{10.11}$$

and

$$\underline{\mathrm{v}}_{21} = \sum_{\alpha=1}^{E} (\underline{\underline{A}}^{\alpha})^{-1} \underline{\underline{a}} \underline{\underline{R}} \underline{\mathrm{v}} \text{ together with } \underline{\mathrm{v}}_{22} = \sum_{\alpha=1}^{E} (\underline{\underline{A}}^{\alpha})^{-1} \underline{\underline{j}} \underline{\underline{R}} \underline{\mathrm{v}} \tag{10.12}$$

Similarly, Eq. (9.20) is:

$$\underline{u}_{21} = \sum_{\alpha=1}^{E} (\underline{\underline{A}}^{\alpha})^{-1} \left(\underline{\tilde{f}}_{\Delta} - \left[\!\left[\underline{\underline{R}} \right]\!\right] \tilde{\underline{u}}_{P} \right) \text{ and } (\underline{u}_{11})_{\Pi} = \sum_{\alpha=1}^{E} \left(\underline{\underline{A}}_{\Pi\Pi}^{\alpha}\right)^{-1} \underline{\underline{A}}_{\Pi\Delta} \underline{\underline{j}} \tilde{\underline{u}}_{P} \tag{10.13}$$

Finally, we recall once more that in view of Eq. (10.10) computation of the components of a harmonic vector requires calculation of local inverses, exclusively.

XI. SCHUR-COMPLEMENT METHOD OF SOLUTION FOR DUAL-PRIMAL FORMULATIONS

Here, the case when not all primal nodes are interior will be considered. The subset of primal nodes that are not interior will be $\pi \subset \Pi$ and we write $\mathrm{I} \subset \Pi$ for the set of interior nodes. Then

$$\Pi = \pi \cup \mathrm{I} \text{ and } \phi = \pi \cap \mathrm{I} \tag{11.1}$$

Furthermore, we write

$$\Gamma \equiv \Delta \tag{11.2}$$

And we will use an obvious notation similar to that defined by Eq. (3.3). Eqs. (3.2) and (3.4) yield

$$\underline{\underline{A}} = \begin{pmatrix} \underline{\underline{A}}_{II} & \underline{\underline{A}}_{I\pi} & \underline{\underline{A}}_{I\Delta} \\ \underline{\underline{A}}_{\pi I} & \underline{\underline{A}}_{\pi\pi} & \underline{\underline{A}}_{\pi\Delta} \\ \underline{\underline{A}}_{\Delta I} & \underline{\underline{A}}_{\Delta\pi} & \underline{\underline{A}}_{\Delta\Delta} \end{pmatrix}, \quad \underline{\underline{L}} = \begin{pmatrix} \underline{\underline{A}}_{II} & \underline{\underline{A}}_{I\pi} & \underline{\underline{A}}_{I\Delta} \\ \underline{\underline{A}}_{\pi I} & \underline{\underline{A}}_{\pi\pi} & \underline{\underline{A}}_{\pi\Delta} \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and } \underline{\underline{R}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \underline{\underline{A}}_{\Delta I} & \underline{\underline{A}}_{\Delta\pi} & \underline{\underline{A}}_{\Delta\Delta} \end{pmatrix} \tag{11.3}$$

We notice also that

$$\underline{\underline{A}}_{\Pi\Pi} = \begin{pmatrix} \underline{\underline{A}}_{II} & \underline{\underline{A}}_{I\pi} & 0 \\ \underline{\underline{A}}_{\pi I} & \underline{\underline{A}}_{\pi\pi} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{11.4}$$

Given $\underline{\mathrm{v}} \in D$, its components $\underline{\mathrm{v}}_{11}$ and $\underline{\mathrm{v}}_{12}$ are given by the two systems of Eq. (9.5). They are

$$\begin{pmatrix} \underline{\underline{A}}_{II} & \underline{\underline{A}}_{I\pi} \\ \underline{\underline{A}}_{\pi I} & \underline{\underline{A}}_{\pi\pi} \end{pmatrix} \begin{pmatrix} (\underline{\mathrm{v}}_{11})_I \\ (\underline{\mathrm{v}}_{11})_\pi \end{pmatrix} = - \begin{pmatrix} \underline{\underline{A}}_{I\Delta} \underline{\underline{j}} \underline{\mathrm{v}}_\Delta \\ \underline{\underline{A}}_{\pi\Delta} \underline{\underline{j}} \underline{\mathrm{v}}_\Delta \end{pmatrix} \tag{11.5}$$

and

$$\begin{pmatrix} \underline{\underline{A}}_{II} & A_{I\pi} \\ \underline{\underline{A}}_{\pi I} & A_{\pi\pi} \end{pmatrix} \begin{pmatrix} (\underline{\mathrm{v}}_{12})_I \\ (\underline{\mathrm{v}}_{12})_\pi \end{pmatrix} = - \begin{pmatrix} \underline{\underline{A}}_{I\Delta} \underline{\underline{a}} \underline{\mathrm{v}}_\Delta \\ \underline{\underline{A}}_{\pi\Delta} \underline{\underline{a}} \underline{\mathrm{v}}_\Delta \end{pmatrix}, \tag{11.6}$$

respectively. When the cardinality of π is much smaller than that associated with I, an efficient manner of solving these systems is by applying a standard Schur complement approach to the matrix of the left-hand sides of these equations. This yields $\underline{\mathrm{v}}_{11}$ and $\underline{\mathrm{v}}_{12}$.

The components $\underline{\mathrm{v}}_{21}$ and $\underline{\mathrm{v}}_{22}$, on the other hand, are given by Eqs. (9.16) and (9.17). They are

$$\underline{\underline{A}} \underline{\mathrm{v}}_{21} = \underline{\underline{a}} \underline{\underline{R}} \underline{\mathrm{v}} \text{ and } \underline{\underline{A}} \underline{\mathrm{v}}_{22} = \underline{\underline{j}} \underline{\underline{R}} \underline{\mathrm{v}} \tag{11.7}$$

We define the restricted average and restricted jump, as those which take the average and the jump, only at the primal degrees of freedom; they will be denoted by $\bar{\underline{\underline{a}}}$ and $\bar{\underline{\underline{j}}}$, respectively. When primal-dual preconditioners are used, we require that $\bar{\underline{\underline{j}}} \underline{u}_{21} = 0$ and $\bar{\underline{\underline{j}}} \underline{u}_{22} = 0$. By reordering, we write each vector $\underline{w} \in D \subset \tilde{D}(\bar{\Omega})$ in the form

$$\underline{w} \equiv \begin{pmatrix} \underline{w}_I \\ \underline{w}_\pi \\ \underline{w}_\Gamma \end{pmatrix} \rightarrow \underline{w} \equiv \begin{pmatrix} \underline{w}_J \\ \underline{w}_\pi \end{pmatrix} \tag{11.8}$$

where

$$\underline{w}_J \equiv \begin{pmatrix} \underline{w}_I \\ \underline{w}_\Gamma \end{pmatrix} \tag{11.9}$$

Then

$$\underline{\underline{A}} \underline{w} \rightarrow \underline{\underline{Q}} \underline{w} \tag{11.10}$$

with

$$\underline{\underline{Q}}^i = \begin{pmatrix} \underline{\underline{Q}}^i_{JJ} & \underline{\underline{Q}}^i_{J\pi} \\ \underline{\underline{Q}}^i_{\pi J} & \underline{\underline{Q}}^i_{\pi\pi} \end{pmatrix}; \quad i = 1, \ldots, E \tag{11.11}$$

and

$$\underline{\underline{Q}} \equiv \sum_{i=1}^{E} \underline{\underline{Q}}^i \tag{11.12}$$

Then, we define the local Schur complement by

$$\underline{\underline{S}}^i = \underline{\underline{Q}}^i_{\pi\pi} - \underline{\underline{Q}}^i_{\pi J}\left(\underline{\underline{Q}}^i_{JJ}\right)^{-1}\underline{\underline{Q}}^i_{J\pi}; i = 1, \ldots, E \tag{11.13}$$

and

$$\underline{\underline{S}} \equiv \sum_{i=1}^{E} \underline{\underline{S}}^i \tag{11.14}$$

It can be seen that the Schur complement is positive definite in the subspace where $\bar{\underline{\underline{j}}}\underline{w} = 0$. Furthermore, Eq.(11.7) can be expressed in terms of the Schur complement, and then the conjugate gradient method can be applied.

XII. IMPLEMENTATION AND COMPUTATIONAL RESULTS

This article was initially intended as a continuation of the work started by Herrera in [13], where the round-trip algorithm was first introduced; thus, only the two-level methods derived from it were numerically tested. For this purpose, the two-dimensional unit square $\Omega = (0, 1)^2$ was used. The square was decomposed into N^2 subdomains having sides of length $\frac{1}{N}$. Each such subdomain was further divided into M^2 smaller squares with sides of length $\frac{1}{M \cdot N}$. As mentioned in this article's text, the conjugate gradient algorithm was used — first without a preconditioner and then applied with a dual-primal preconditioner and then again with a preconditoner under study: the "projecion preconditioner". The stopping criteria was a reduction of the residue to less than 10^{-6}. The problems tested involved elliptic operators that were strictly positive definite on all subdomains. For the case of the Laplacian operator, the dual-primal preconditioned method ensures nonsingular matrices on all subdomains and that is the only kind of preconditioners that has been so far utilized.

In [13], only the case when all primal nodes are interior was considered. Then, when that approach is applied, the choice of $\tilde{D}_2(\Delta)$ is unique and consists of continuous functions exclusively. On the other hand, there are many choices of $\tilde{D}_1(\Delta)$, which satisfy

$$\tilde{D}(\Delta) = \tilde{D}_1(\Delta) \oplus \tilde{D}_2(\Delta) \tag{12.1}$$

However, when the approach here presented is applied the choices for $\tilde{D}_1(\Delta)$ and $\tilde{D}_2(\Delta)$ are unique; they are defined by

$$\tilde{D}_1(\Delta) \equiv \underline{\underline{j}}\tilde{D}(\Delta) \text{ and } \tilde{D}_2(\Delta) \equiv \underline{\underline{a}}\tilde{D}(\Delta) \tag{12.2}$$

That is, $\tilde{D}_1(\Delta)$ is the Euclidean orthogonal complement with respect to $\tilde{D}(\Delta)$ of the continuous functions contained in $\tilde{D}(\Delta)$. In our numerical experiments, carried out for this article, the method was first tested using the approach and basis functions of [13] and subsequently updated to incorporate the new procedure, together with the average $\underline{\underline{a}}$ and jump $\underline{\underline{j}}$ matrices. Then, the number of iterations dropped significantly to about one half of the previous values. This would suggest that these matrices, being, symmetric, nonnegative and orthogonal projection-matrices, have superior computational properties and that the $\underline{\underline{j}}$ operator is the optimal choice for the $\boldsymbol{B}$ operator of the FETI methods [11]. To test this hypothesis, Antonio Carrilllo, a PhD student working in a project

TABLE I. Comparison of $\boldsymbol{B}$ versus $\underline{\underline{j}}$.

Subdomains	Total nodes	Iterations FETI with $\boldsymbol{B}$	Iterations FETI with $\underline{\underline{j}}$
169	28,224	2,323	102
256	57,600	2,626	135
324	104,329	2,941	125
400	159,201	3,618	168

whose goal is to thoroughly compare the FETI methods with the multipliers-free methods, carried out the computations that are described next. He applied the one-level FETI method, not using any preconditioner, first using the $\boldsymbol{B}$ operator of the FETI methods described in [11] and then he replaced the $\boldsymbol{B}$ operator by our matrix $\underline{\underline{j}}$, introduced in this article. Of course the one-level FETI method, when it is not preconditioned, is not a competitive method, but the purpose of the experiment was only to test the effect of replacing the $\boldsymbol{B}$ operator by our matrix $\underline{\underline{j}}$, and in this respect the result of this experiment is very significant: the replacement of the $\boldsymbol{B}$ operator by the matrix $\underline{\underline{j}}$ reduced the number of iterations to essentially only 5%; a saving of 95%. A complete list of the results of these numerical experiments is given in Table I.

A second observation is that the version of dual-primal methods introduced in this article requires computations involving only $\underline{\underline{L}}$, $\underline{\underline{R}}$, $\underline{\underline{A}}^{-1}_{\alpha\Pi\Pi}$, and $\underline{a}$, leading to a straightforward implementation that, furthermore, does not require the colateral Lagrange multipliers, with a concomitant reduction in the degrees of freedom. However, a publication is being prepared in which this point is discussed more thoroughly. From the results that have been obtained so far, of the preconditioners that were tested, the dual-primal method is clearly the best—both in number of iterations as well as in computer time. It can also be noted that in comparing the Round-trip operators $(I + T_D)$ and $(I + T_N)$, the $(I + T_D)$ operator gives somewhat better results. Another positive feature of the new approach is that the same code works for all variants of the two-level methods.

XIII. DISCUSSION AND CONCLUSIONS

This article deals with nonoverlapping domain decomposition methods and for them, it introduces a unified theory that incorporates from the start the possibility of including dual-primal preconditioners; then, the different preconditioners reported in the literature can be derived by simply choosing different dual subspaces. The entire theory is derived without recourse to Lagrange multipliers.

The starting point is a general setting with the conspicuous feature that it incorporates dual-primal preconditioners ab initio. In such a setting, one-level and two-level methods have been developed in an integrated manner. Among one-level methods, a general Dirichlet and a general Neumann approach are presented; the Dirichlet approach yields the standard Schur complement formulation, whereas the Neumann approach yields a formulation that is very similar, probably equivalent, to the one-level FETI formulation. Two-level methods are introduced by means of a single general algorithm, the "round-trip" algorithm, which was first presented by Herrera in [13] and is here developed at the matrix level. They include methods that are alike to the Neumann-Neumann and to the preconditioned FETI (Dirichlet-Dirichlet, according to Toselli and Widlund [11]). All these procedures and the matrix expressions here developed for them can, in a direct manner, be applied indistinctly with (or without) dual-primal preconditioners.

As mentioned earlier, different choices of the dual subspaces yield the different dual-primal preconditioners reported in the literature.

Some contributions of this article and the unified theory it contains that deserve to be mentioned are as follows:

1. The significant simplification of the computational codes required for the algorithms implementation.
2. The algorithms are derived directly from the problem-matrices, independently of the partial differential equations that originated them, and the number of dimensions of the problem; thus, codes developed for its application to 2D-problems can be easily modified for its application to 3D-problems. In standard treatments, the dimensions of the space are defined from the start.
3. The matrices $\underline{\underline{a}}$ and $\underline{\underline{j}}$, which are also a contribution contained in this article, exhibit superior computational properties in the numerical experiments. In particular, the $\underline{\underline{j}}$ operator seems to be the optimal choice for the $\boldsymbol{B}$ operator of the FETI methods [11]. In numerical experiments discussed in the section on Computational Results, very significant reductions in the number of iterations were achieved when the matrix $\underline{\underline{j}}$ was used instead of standard $\boldsymbol{B}$ operators.
4. These matrices are generalizations of the "average" and "jump" of a function, which can be effectively applied at the discrete level (i.e., to vectors) not only at internal-boundary nodes but at edges and corners, as well. They are symmetric and non-negative projection matrices on complementary subspaces. Furthermore, their construction is very simple; indeed, $\underline{\underline{a}}$ is the average over each node and, once $\underline{\underline{a}}$ is available, $\underline{\underline{j}}$ derives from it. Another useful property is $\underline{\underline{a}}\,\underline{\underline{j}} = \underline{\underline{j}}\,\underline{\underline{a}} = \underline{\underline{0}}$.
5. Elimination of Lagrange multipliers in the formulation of the methods yields significant reductions in the number of degrees of freedom in many problems.
6. In section XI, an efficient method of solution applicable to general dual-primal preconditioned formulations is introduced.
7. The unified theory implies a new expression for the *Steklov-Poincaré* operator (i.e, the jump of the normal derivative) at the discrete level. A significant difference of the new expression, with respect to standard formulas, lies in the fact that it does not contain the right-hand side of the equation to be solved (compare, for example, with [9] and [11], pp 3 and 4); winning thereby, in theoretical consistency.
8. This article contains also numerical experiments that use competitive preconditioners, which incorporate a coarse space.
9. Using our unified theory we have also derived a class of preconditioners, projection-preconditioners, which have some attractive features and are now under study.

XIV. APPENDIX

Derivation of the Average and Jump Matrices

This Appendix is devoted to derive explicit expressions for the matrices $\underline{\underline{a}}$ and $\underline{\underline{j}}$ introduced and extensively used in the text of the article.

Consider the space $\mathcal{R}^m$, of vectors of dimension m, provided with the Euclidean inner product. Then, $\mathcal{R}^m$ can be decomposed into two orthogonal (with respect to the Euclidean inner product)

subspaces

$$\mathcal{R}^m = \mathcal{R}^m_a + \mathcal{R}^m_j \text{ and } \mathcal{R}^m_a \perp \mathcal{R}^m_j \tag{A.1}$$

They are defined by

$$\mathcal{R}^m_a \equiv \{\underline{b} \equiv (b_1, \ldots, b_m) | b_1 = \cdots = b_m\} \text{ and } \mathcal{R}^m_j \equiv \left\{ \underline{b} \equiv (b_1, \ldots, b_m) \Big| \sum_{j=1}^{m} b_j = 0 \right\} \tag{A.2}$$

Then every vector $\underline{b} \equiv (b_1, \ldots, b_d) \in \mathcal{R}^d$ can be written, in a unique manner, as

$$\underline{b} = \underline{b}_a + \underline{b}_j; \quad \text{where } \underline{b}_a \in \mathcal{R}^m_a \text{ and } \underline{b}_j \in \mathcal{R}^m_j \tag{A.3}$$

Explicit expressions for the vectors $\underline{b}_a \equiv (b_{a1}, \ldots, b_{ad})$ and $\underline{b}_j \equiv (b_{j1}, \ldots, b_{jd})$ are

$$b_{a\alpha} = \frac{1}{d} \sum_{\beta=1}^{d} b_{a\beta}, \quad \alpha = 1, \ldots, d \text{ and } \underline{b}_j = \underline{b} - \underline{b}_a \tag{A.4}$$

The matrices $\underline{\underline{a}}^d$ and $\underline{\underline{j}}^d$, defined by

$$\underline{\underline{a}}^d \equiv d^{-1} \begin{pmatrix} 11\ldots1 \\ 11\ldots1 \\ \ldots\ldots\ldots \\ 11\ldots1 \end{pmatrix} \text{ and } \underline{\underline{j}}^d \equiv \begin{pmatrix} 10\ldots0 \\ 01\ldots0 \\ \ldots\ldots\ldots \\ 00\ldots1 \end{pmatrix} - d^{-1} \begin{pmatrix} 11\ldots1 \\ 11\ldots1 \\ \ldots\ldots\ldots \\ 11\ldots1 \end{pmatrix} \tag{A.5}$$

have the property that for every $\underline{b} \equiv (b_1, \ldots, b_m) \in \mathcal{R}^m$ one has

$$\underline{b}_a = \underline{\underline{a}}^m \underline{b} \text{ and } \underline{b}_j = \underline{\underline{j}}^m \underline{b} \tag{A.6}$$

Each one of the matrices $\underline{\underline{a}}^d$ and $\underline{\underline{j}}^d$ is a projection matrix and as such they are symmetric, nonnegative, and idempotent.

Going back to the material contained in Section II, we observe that

$$\tilde{D}(\bar{\Omega}) = \tilde{D}(\mathrm{Z}_1) \oplus \cdots \oplus \tilde{D}(\mathrm{Z}_N) \tag{A.7}$$

where N is number of nodes (i.e., the cardinality of the partition $\mathcal{P}$). Then,

$$\underline{\underline{a}} = \underline{\underline{a}}_1^{|\mathrm{Z}_1|} + \cdots + \underline{\underline{a}}_N^{|\mathrm{Z}_N|} \tag{A.8}$$

Let us write $\mathcal{P} = \mathcal{P}_\Pi \cup \mathcal{P}_\Delta$, where P_Π is the set of primal nodes and $\mathcal{P}_\Delta$ is the set of dual nodes. Then, to finish we observe that

$$\underline{\underline{a}}_\alpha^{|\mathrm{Z}_\alpha|} = \underline{\underline{I}}, \quad \text{whenever } \alpha \in \mathcal{P}_\Pi \tag{A.9}$$

Therefore, Eq. (A.4) needs to be applied only at dual nodes.

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