The Multipliers-Free Domain Decomposition Methods

Ismael Herrera, Robert A. Yates*
Instituto de Geofísica, Universidad Nacional Autónoma de México (UNAM), México City 14000 D.F., Mexico

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This article concludes the development and summarizes a new approach to dual-primal domain decomposition methods (DDM), generally referred to as "the multipliers-free dual-primal method." Contrary to standard approaches, these new dual-primal methods are formulated without recourse to Lagrange-multipliers. In this manner, simple and unified matrix-expressions, which include the most important dual-primal methods that exist at present are obtained, which can be effectively applied to floating subdomains, as well. The derivation of such general matrix-formulas is independent of the partial differential equations that originate them and of the number of dimensions of the problem. This yields robust and easy-to-construct computer codes. In particular, $2D$ codes can be easily transformed into $3D$ codes. The systematic use of the average and jump matrices, which are introduced in this approach as generalizations of the "average" and "jump" of a function, can be effectively applied not only at internal-boundary-nodes but also at edges and corners. Their use yields significant advantages because of their superior algebraic and computational properties. Furthermore, it is shown that some well-known difficulties that occur when primal nodes are introduced are efficiently handled by the multipliers-free dual-primal method. The concept of the Steklov–Poincaré operator for matrices is revised by our theory and a new version of it, which has clear advantages over standard definitions, is given. Extensive numerical experiments that confirm the efficiency of the multipliers-free dual-primal methods are also reported here. Numer Methods Partial Differential Eq 000: 000–000, 2009

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I. INTRODUCTION

Nowadays, parallel computing is the most effective means for increasing computational speed. In turn, DDM is most efficient for applying parallel-computing to the solution of partial differential equations. Since the time when the international community began to intensively study DDM, attention has shifted from overlapping to nonoverlapping methods, mainly because they

Correspondence to: Ismael Herrera, Instituto de Geofísica, Universidad Nacional Autónoma de México (UNAM), México city 14000 D.F., Mexico (e-mail: iherrera@servidor.unam.mx)
*Present address: Alternativas en Computación, S.A. de C.V.
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are more effective for many problems [1]. In particular, it is easier to apply parallel computers to them and to develop the corresponding codes. A direct application of such an approach yields the Schur-complement method, which corresponds to formulating Dirichlet problems in each one of the subdomains, and another one, which corresponds to formulating Neumann problems: the nonpreconditioned FETI method. The performance of these methods, however, usually is not satisfactory and can be drastically improved by applying appropriate preconditioners [2, 3].

Some of the most efficient preconditioned nonoverlapping methods are obtained by using the Neumann method as a preconditioner of the Schur complement method; or, conversely, using the Schur complement method as a preconditioner of the Neumann method. This gives rise to a class of methods that in this article is generically called the "round-trip methods," since one goes from the space of continuous functions to another space having continuous normal derivatives and back; or the same, but in reverse order. When the Schur complement method is preconditioned with the Neumann method, a procedure that is known in the scientific literature as the Neumann–Neumann method is obtained, while the preconditioned FETI is obtained when the Neumann method is preconditioned with the Schur complement method. For a thorough and well-documented review of these methods, the reader is referred to [2], where a broad bibliography is revised.

More recently, the dual-primal FETI methods were introduced [4], in which a relatively small number of continuity constraints across the interfaces are enforced, and they have been very successful because they enhance in a rather significant manner the condition number of the matrices involved in the problems and accelerate the convergence rates.

The treatment of round-trip algorithms, until recently, had been done with recourse to Lagrange multipliers exclusively. However, Herrera and Yates, in a sequence of articles, have introduced a multipliers-free formulation of FETI and dual-primal methods [5–7]. Actually, the whole series constitutes a "general theory of partial differential operators acting on discontinuous functions and of matrices acting on discontinuous vectors."

The theory of partial differential operators acting on discontinuous functions is based on extensive previous work that culminated, and was presented in an integrated form, in the first paper of the series we are referring to [5] where the interested reader can find further references to it. In the second article [6], the general ideas and the theoretical background that permits formulating a very general class of substructuring methods without recourse to Lagrange multipliers were introduced. The results of that article were developed working in finite-dimensional function spaces. On the contrary, the third article introduced an approach that permits carrying out the domain decomposition directly in the matrices themselves, independently of the function-spaces from which they derive [7]. This is a conspicuous feature of the methodology there introduced, which also extended the procedures to include dual-primal methods and developed the matrices in a more explicit manner.

In the present fourth article of the series, we continue working directly on the matrices, using the approach introduced in the third article [7]. The contributions contained herein are multiple; the following should be highlighted:

1. Explicit matrix-formulas are developed for the algorithms, which are expressed in terms of the Schur complement matrices exclusively and, therefore, in terms of the internal-boundary-node value of the vectors involved.
2. The generality of such formulas permits unifying the formulation of the different methods, which are classified broadly into one-way (Schur complement and nonpreconditioned FETI methods) and round-trip methods (Neumann–Neumann and preconditioned FETI).

3. There is a significant simplification in the development of the codes that are required for the implementation of the algorithms. The algorithms are derived directly from the problem-matrices, independently of the partial differential equations that originated them and the number of dimensions of the problem.
4. The robustness of the codes so obtained. When the matrix-formulas here supplied are used, by simple substitutions parallel-processing codes can be developed for any system that is governed by linear differential equations, or by systems of such equations, that are symmetric and non-negative. For example, for the numerical experiments presented in Section XVI, $2D$ codes were easily transformed into $3D$ codes. This property is, to a large extent, due to the fact that the derivations of such matrix-formulas are independent of the problems from which they stem. In standard treatments the space-dimension is defined from the start.
5. The average and jump matrices $\underline{\underline{a}}$ and $\underline{\underline{j}}$, respectively, which were introduced in [7] and have been used extensively, exhibit superior algebraic and computational properties. In particular, the $\underline{\underline{j}}$ operator is the optimal choice for the $\boldsymbol{B}$ operator of the FETI methods [2]. In numerical experiments that have been carried out, very significant reductions in the number of iterations were achieved when the matrix $\underline{\underline{j}}$ was used instead of standard $\boldsymbol{B}$ operators [7]. These matrices are generalizations of the "average" and "jump" of a function, which can be effectively applied at the discrete level (i.e., to vectors), not only at internal-boundary-nodes but also at edges and corners.
6. Very effective means for treating floating subdomains are developed in this article. This is especially relevant, since there are applications of great importance in which they occur, as in problems that are formulated in terms of Laplace, biharmonic, or static-elasticity operators.
7. As it is well known, the parallel implementation of dual-primal methods is impaired by the introduction of primal-nodes. When the multipliers-free formulation of this series of papers is applied, such a handicap is explicitly expressed and procedures for reducing it to a minimum are given (see Appendix D).
8. Extensive numerical results that confirm experimentally the efficiency of the multipliers-free dual-primal methods are also reported here.
9. Finally, the multipliers-free formulation of this kind of methods implies a new interpretation of the Steklov–Poincaré operator for matrices that is presented here, in Section IX.

Although this article is based on developments introduced in the previous articles of the series [5–7], a significant number of modifications and new theoretical developments were required, which are presented in Sections II to VI. The multipliers-free formulation of the general problem here treated is first introduced in Section VII. The Green–Herrera formulas, which have had a significant role in theory of partial differential operators acting on discontinuous functions, are extended to matrices, in Section VIII. Then, a new interpretation of Steklov–Poincaré operator emerges from this extension of Green–Herrera formula, which is given in Section IX. A formulation of the problem treated in terms of Schur-complement matrices exclusively, is given in Section XI, while Section X is preparatory to it. One-way and round-trip algorithms are discussed in Sections XII and XIII, respectively. The procedure for effectively treating floating subdomains, which is based on a generalization of the Schur-complement matrix, is introduced in Section XIV, while Section XV is devoted to explain an effective procedure for inverting such a generalized Schur-complement matrix. Section XVI is devoted to implementation issues and numerical results, while the conclusions of this article are summarized in Section XVII. Several details that are required for the implementation of the algorithms are presented in the Appendices A to D.

II. NODES AND THEIR CLASSIFICATION

Let the set of "original nodes" be $\Omega \equiv \{1, \ldots, d\}$, while the family $\{\Omega_1, \ldots, \Omega_E\} \subset \Omega$ is a cover of Ω; i.e.,

$$\Omega = \bigcup_{\alpha=1}^{E} \Omega_\alpha \tag{2.1}$$

We also consider pairs $\underline{p} \equiv (p, \alpha)$, such that $p \in \Omega$ and $\alpha \in \{1, \ldots, E\}$. Then, we define

$$\bar{\Omega} \equiv \left\{ \underline{p} = (p, \alpha) \middle| p \in \Omega_\alpha \right\} \tag{2.2}$$

And, for every $p \in \Omega$,

$$\mathrm{Z}(p) \equiv \{\alpha \in \{1, \ldots, E\} | (p, \alpha) \in \bar{\Omega}\} \tag{2.3}$$

while the multiplicity of p, $m(p)$, will be the cardinality of $\mathrm{Z}(p)$. The pairs $\underline{p} \equiv (p, \alpha)$ that belong to $\bar{\Omega}$ are said to be "derived nodes." We distinguish two classes of original nodes: when $m(p) = 1$, $p \in \Omega$ is said to be an "interior original node" and when $m(p) > 1$, it is said to be a "(internal-)boundary original node"; the sets of interior original nodes and boundary original nodes, which are disjoint, will be denoted by Ω^{I} and Ω^{Γ}, respectively. Similarly, we distinguish two classes of derived nodes: $\underline{p} \equiv (p, \alpha) \in \bar{\Omega}$ is said to be an interior (derived) node, or a boundary (derived) node, depending on whether the multiplicity of p is equal or greater than 1, respectively. The sets of interior (derived) nodes and of boundary (derived) nodes will be denoted by I and Γ, respectively.

We choose a set $\Omega^{\pi} \subset \Omega^{\Gamma}$ and define the sets:

$$\begin{cases} \mathrm{I} \equiv \left\{ \underline{p} = (p, \alpha) \in \bar{\Omega} | p \in \Omega^{I} \right\} \\ \pi \equiv \left\{ \underline{p} = (p, \alpha) \in \bar{\Omega} | p \in \Omega^{\pi} \right\} \\ \Delta \equiv \left\{ \underline{p} = (p, \alpha) \in \bar{\Omega} | p \in \Omega^{\Gamma} - \Omega^{\pi} \right\} \end{cases} \tag{2.4}$$

Clearly, $\Delta = \Gamma - \pi$. Furthermore, we define $\Pi \equiv \mathrm{I} \cup \pi$ and observe that

$$\bar{\Omega} = \mathrm{I} \cup \pi \cup \Delta = \Pi \cup \Delta \text{ and } \emptyset = \Pi \cap \Delta = \pi \cap \Delta = \pi \cap \mathrm{I} = \Delta \cap \mathrm{I} \tag{2.5}$$

The sets π and Δ are the sets of primal and "dual" (derived) nodes, respectively. We observe that $\Delta = \Gamma$, when $\pi = \emptyset$.

III. VECTORS AND CONTINUOUS VECTORS

Notice that every real-valued function defined either in Ω or in $\bar{\Omega}$ is a vector (so, they will be referred to indistinctly as functions or vectors). The linear spaces $\tilde{D}(\Omega)$ and $\tilde{D}(\bar{\Omega})$ will be constituted by the functions (vectors) defined in Ω and in $\bar{\Omega}$, respectively. Similarly, $\tilde{D}(\Pi) \subset \tilde{D}(\bar{\Omega})$ and $\tilde{D}(\Delta) \subset \tilde{D}(\bar{\Omega})$ will be the linear subspaces of $\tilde{D}(\bar{\Omega})$ whose elements vanish outside Π and Δ, respectively. The subspaces $\tilde{D}(\mathrm{I})$, $\tilde{D}(\pi)$, and $\tilde{D}(\Gamma)$, of $\tilde{D}(\bar{\Omega})$, are defined similarly. Then,

$$\tilde{D}(\bar{\Omega}) = \tilde{D}(\Pi) \oplus \tilde{D}(\Delta) \tag{3.1}$$

Here, and in what follows, the symbol $\oplus$ stands for the direct sum of two linear spaces; thus, Eq. (3.1) is fulfilled if and only if

$$\begin{cases} \tilde{D}(\bar{\Omega}) = \tilde{D}(\Pi) + \tilde{D}(\Delta) \\ \{0\} \quad = \tilde{D}(\Pi) \cap \tilde{D}(\Delta) \end{cases} \tag{3.2}$$

Therefore, vectors of $\tilde{D}(\bar{\Omega})$ can be uniquely represented as

$$\underline{u} = (\underline{u}_\Pi, \underline{u}_\Delta) = \underline{u}_\Pi + \underline{u}_\Delta, \quad \text{with } \underline{u}_\Pi \in \tilde{D}(\Pi) \text{ and } \underline{u}_\Delta \in \tilde{D}(\Delta) \tag{3.3}$$

The natural immersion of $\tilde{D}(\Omega)$ into $\tilde{D}(\bar{\Omega})$, denoted by $\tau : \tilde{D}(\Omega) \to \tilde{D}(\bar{\Omega})$, is defined for every $\widehat{\underline{u}} \in \tilde{D}(\Omega)$ by

$$\left(\tau \widehat{\underline{u}}\right)\left(\underline{q}\right) = \widehat{\underline{u}}(p), \quad \forall \underline{q} \in \mathrm{Z}(p) \subset \bar{\Omega} \tag{3.4}$$

More explicitly, this yields

$$\left(\tau \widehat{\underline{u}}\right)_{(p,\alpha)} = \widehat{\underline{u}}(p), \quad \forall (p,\alpha) \in \bar{\Omega} \tag{3.5}$$

The image $\tau \tilde{D}(\Omega)$ of $\tilde{D}(\Omega)$ under $\tau : \tilde{D}(\Omega) \to \tilde{D}(\bar{\Omega})$ constitutes a linear subspace of $\tilde{D}(\bar{\Omega})$. We define

$$\bar{D}(\bar{\Omega}) \equiv \tau \tilde{D}(\Omega) \subset \tilde{D}(\bar{\Omega}) \tag{3.6}$$

Vectors belonging to $\bar{D}(\bar{\Omega})$ will be said to be "continuous vectors." According to Eq. (3.5), continuous vectors are characterized by the fact that their value at any derived node, (p,α), is independent of α and only depends on the original node, p, from which it derives. Clearly, the mapping $\tau : \tilde{D}(\Omega) \to \bar{D}(\bar{\Omega})$ is a bijection; this permits defining $\tau^{-1} : \bar{D}(\bar{\Omega}) \to \tilde{D}(\Omega)$, which has the property that, for any $\underline{u} \in \bar{D}(\bar{\Omega})$, one has

$$\left(\tau^{-1}\underline{u}\right)(p) = \underline{u}(p,\alpha), \quad \forall p \in \Omega \text{ and } \alpha \in \mathrm{Z}(p) \tag{3.7}$$

IV. THE EUCLIDEAN INNER PRODUCTS

The "Euclidean inner product," which is the only one to be considered in the first part of this article, is defined to be

$$\begin{cases} \widehat{\underline{u}} \bullet \widehat{\underline{w}} \equiv \sum_{p\in\Omega} \widehat{\underline{u}}(p)\widehat{\underline{w}}(p), \quad \forall \widehat{\underline{u}},\ \widehat{\underline{w}} \in \tilde{D}(\Omega) \\ \underline{u} \bullet \underline{w} \equiv \sum_{\underline{p}\in\bar{\Omega}} \underline{u}(\underline{p})\underline{w}(\underline{p}) = \sum_{q\in\Omega} \sum_{\underline{p}\in\mathrm{Z}(q)} u(\underline{p})w(\underline{p}), \quad \forall \underline{u},\underline{w} \in \tilde{D}(\bar{\Omega}) \end{cases} \tag{4.1}$$

The methods described in this article are not restricted, in their applicability, to a single differential equation, but they are equally applicable to systems of differential equations, such as those occurring in elasticity. A proper treatment in our scheme of those systems requires introducing vector-valued functions. In such cases, $\widehat{\underline{u}}(p)$ and $\underline{u}(\underline{p})$ are themselves vectors and, when defining the Euclidean inner product, Eq. (4.1) must be replaced by

$$\begin{cases} \widehat{\underline{u}} \bullet \widehat{\underline{w}} \equiv \sum_{p\in\Omega} \widehat{\underline{u}}(p) \odot \widehat{\underline{w}}(p), \quad \forall \widehat{\underline{u}},\ \widehat{\underline{w}} \in \tilde{D}(\Omega) \\ \underline{u} \bullet \underline{w} \equiv \sum_{\underline{p}\in\bar{\Omega}} \underline{u}(\underline{p}) \odot \underline{w}(\underline{p}) = \sum_{q\in\Omega} \sum_{\underline{p}\in\mathrm{Z}(q)} u(\underline{p}) \odot w(\underline{p}), \quad \forall \underline{u},\ \underline{w} \in \tilde{D}(\bar{\Omega}) \end{cases} \tag{4.2}$$

Here, the symbol $\odot$ stands for the inner product of the vector space where the vectors $\widehat{\underline{u}}(p)$ and $\underline{u}(\underline{p})$ lie.

Two auxiliary matrices are introduced next; they are: $\widehat{\underline{\underline{m}}} : \tilde{D}(\Omega) \to \tilde{D}(\Omega)$ and $\underline{\underline{m}} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$, which are defined, for each $\widehat{\underline{u}} \in \tilde{D}(\Omega)$ and each $\underline{u} \in \tilde{D}(\bar{\Omega})$, by

$$\begin{aligned} \widehat{\underline{\underline{m}}}\widehat{\underline{u}}(p) &= m(p)\widehat{\underline{u}}(p), \quad \forall p \in \Omega \\ \underline{\underline{m}}\underline{u}(\underline{p}) &= m(p)\underline{u}(\underline{p}), \quad \forall \underline{p} = (p,\alpha) \in \bar{\Omega} \end{aligned} \tag{4.3}$$

Both of them are diagonal matrices. The values at the main diagonals of $\widehat{\underline{\underline{m}}}$ and $\underline{\underline{m}}$ are the multiplicities $m(\underline{p})$. Simple results whose proofs are straightforward are as follows:

$$\tau\widehat{\underline{\underline{m}}}\widehat{\underline{u}} = \underline{\underline{m}}\tau\widehat{\underline{u}} \text{ and } \tau\widehat{\underline{\underline{m}}}^{-1}\widehat{\underline{u}} = \underline{\underline{m}}^{-1}\tau\widehat{\underline{u}}, \quad \forall\widehat{\underline{u}} \in \tilde{D}(\Omega) \tag{4.4}$$

Together with

$$\underline{\underline{m}}\bar{D}(\Omega) = \bar{D}(\Omega) = \underline{\underline{m}}^{-1}\bar{D}(\Omega) \tag{4.5}$$

Lemma 4.1. *When $\widehat{\underline{u}}, \widehat{\underline{w}} \in \tilde{D}(\Omega)$, each one the following relations holds:*

$$\left.\begin{aligned} \widehat{\underline{u}} \bullet \widehat{\underline{\underline{m}}}\widehat{\underline{w}} &= \tau\big(\widehat{\underline{u}}\big) \bullet \tau\big(\widehat{\underline{w}}\big) \\ \widehat{\underline{u}} \bullet \widehat{\underline{w}} &= \tau\big(\underline{u}\big) \bullet \underline{\underline{m}}^{-1}\tau\big(\underline{w}\big) \end{aligned}\right\}, \quad \forall\widehat{\underline{u}}, \widehat{\underline{w}} \in \tilde{D}(\Omega) \tag{4.6}$$

Proof. We write $\underline{u} \equiv \tau\widehat{\underline{u}}$ and $\underline{w} \equiv \tau\widehat{\underline{w}}$, and then apply Eqs. (4.1) and (4.2), together with Eq. (3.4), to obtain

$$\underline{u} \bullet \underline{w} = \sum_{p\in\Omega}\sum_{\underline{q}\in Z(p)} \underline{u}(\underline{q})\underline{w}(\underline{q}) = \sum_{p\in\Omega} m(p)\widehat{\underline{u}}(p)\widehat{\underline{w}}(p) = \widehat{\underline{u}} \bullet \widehat{\underline{\underline{m}}}\widehat{\underline{w}} \tag{4.7}$$

This shows the first relation of Eq. (4.6). Then, applying such a relation with $\widehat{\underline{w}}$ replaced by $\widehat{\underline{\underline{m}}}^{-1}\widehat{\underline{w}}$ the second one is obtained. ∎

Corollary 4.1. *Let $\underline{u} \in \tau\tilde{D}(\Omega) = \bar{D}(\bar{\Omega})$ be such that for some $\widehat{\underline{u}} \in D(\Omega)$ it fulfills*

$$\widehat{\underline{u}} \bullet \widehat{\underline{w}} = \underline{u} \bullet \tau\big(\widehat{\underline{w}}\big), \quad \forall\widehat{\underline{w}} \in D(\Omega) \tag{4.8}$$

Then

$$\underline{u} = \underline{\underline{m}}^{-1}\tau\big(\widehat{\underline{u}}\big) = \tau\big(\widehat{\underline{\underline{m}}}^{-1}\widehat{\underline{u}}\big) \tag{4.9}$$

Proof. In view of the second relation in Eq. (4.6) one has

$$\widehat{\underline{w}} \bullet \widehat{\underline{u}} = \tau\big(\widehat{\underline{w}}\big) \bullet \underline{\underline{m}}^{-1}\tau\big(\widehat{\underline{u}}\big), \quad \forall\widehat{\underline{w}} \in \tilde{D}(\Omega), \tag{4.10}$$

which implies

$$\big(\underline{u} - \underline{\underline{m}}^{-1}\tau\big(\widehat{\underline{u}}\big)\big) \bullet \underline{w} = 0, \quad \forall\underline{w} \in \tau\tilde{D}(\Omega) = \bar{D}(\Omega) \tag{4.11}$$

Then Corollary (4.1) is clear, since both $\underline{u}$ and $\underline{\underline{m}}^{-1}\tau(\widehat{\underline{u}})$ belong to $\bar{D}(\Omega)$. ∎

V. VECTOR SUBSPACES. THE AVERAGE AND JUMP MATRICES

Two matrices $\underline{\underline{a}} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ and $\underline{\underline{j}} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ are now introduced, which are defined by

$$\underline{\underline{a}}\underline{u} = Proj_{\bar{D}}\underline{u} \text{ and } \underline{\underline{j}} = \underline{\underline{I}} - \underline{\underline{a}} \tag{5.1}$$

Here, $\underline{\underline{I}}$ is the identity matrix and the projection on $\bar{D}$ is taken with respect to the Euclidean inner product. The matrices $\underline{\underline{a}}$ and $\underline{\underline{j}}$ are referred to as the "average and the "jump" matrices. Clearly, $\underline{\underline{I}} = \underline{\underline{a}} + \underline{\underline{j}}$ and

$$\bar{D}(\bar{\Omega}) \equiv \underline{\underline{a}}\tilde{D}(\bar{\Omega}) \tag{5.2}$$

Furthermore, $\underline{\underline{j}}$ is also a projection; indeed, it is the projection on the orthogonal complement of $\bar{D}$; in particular, $\underline{\underline{j}}\bar{D}(\bar{\Omega}) = \{0\}$. The following properties should be noticed: $\underline{\underline{a}}$ and $\underline{\underline{j}}$ are both symmetric, non-negative, and idempotent. Furthermore,

$$\underline{\underline{a}}\,\underline{\underline{j}} = \underline{\underline{j}}\underline{\underline{a}} = \underline{\underline{0}} \tag{5.3}$$

The construction of the matrix $\underline{\underline{a}}$ is relatively simple [previous paper]. Writing

$$\underline{\underline{a}} \equiv \left(a_{(i,\alpha)(j,\beta)}\right) \tag{5.4}$$

Then,

$$a_{(i,\alpha)(j,\beta)} = \frac{1}{m(i)}\delta_{ij}, \quad \forall\alpha \in \mathrm{Z}(i) \text{ and } \forall\beta \in \mathrm{Z}(j) \tag{5.5}$$

An expression for the matrix $\underline{\underline{j}}$ follows from Eq. (5.1), but its action on any vector is easily obtained using:

$$\underline{\underline{j}}\underline{u} = \underline{u} - \underline{\underline{a}}\underline{u}, \quad \forall\underline{u} \in \bar{\Omega} \tag{5.6}$$

The following subspaces are now introduced:

$$\left.\begin{aligned}\tilde{D}_{11}(\bar{\Omega}) &\equiv \underline{\underline{j}}\tilde{D}(\bar{\Omega}) \subset \tilde{D}(\Gamma)\\ \tilde{D}_{12}(\bar{\Omega}) &\equiv \bar{D}(\bar{\Omega}) = \underline{\underline{a}}\tilde{D}(\bar{\Omega})\end{aligned}\right\} \text{ and } \begin{cases}\tilde{D}_{11}(\Gamma) \equiv \underline{\underline{j}}\tilde{D}(\Gamma) = \tilde{D}_{11}(\bar{\Omega})\\ \tilde{D}_{12}(\Gamma) \equiv \underline{\underline{a}}\tilde{D}(\Gamma)\end{cases} \tag{5.7}$$

And the following relations are here highlighted:

$\tilde{D}_{11}(\bar{\Omega}) = \tilde{D}_{11}(\Gamma)$ is the orthogonal complement, with respect to the Euclidean inner product, of $\tilde{D}_{12}(\bar{\Omega})$;

$$\tilde{D}(\bar{\Omega}) = \underline{\underline{a}}\tilde{D}(\bar{\Omega}) \oplus \underline{\underline{j}}\tilde{D}(\bar{\Omega}) = \bar{D}(\bar{\Omega}) \oplus \underline{\underline{j}}\tilde{D}(\bar{\Omega}) = \tilde{D}(\mathrm{I}) \oplus \tilde{D}_{11}(\Gamma) \oplus \tilde{D}_{12}(\Gamma) \tag{5.8}$$

$$\tilde{D}(\bar{\Omega}) = \tilde{D}_{11}(\Gamma) \oplus \bar{D}(\bar{\Omega}) \tag{5.9}$$

$$\tilde{D}(\Gamma) = \tilde{D}_{11}(\Gamma) \oplus \tilde{D}_{12}(\Gamma) \tag{5.10}$$

Furthermore, $\tilde{D}_{11}(\Gamma)$ and $\tilde{D}_{12}(\Gamma)$ are orthogonal complements relative to $\tilde{D}(\Gamma)$;

$$\bar{D}(\bar{\Omega}) = \tilde{D}(\mathrm{I}) \oplus \tilde{D}_{12}(\Gamma) \tag{5.11}$$

$$\underline{\underline{a}}\,\underline{u} = \underline{u} \quad \text{and} \quad \underline{\underline{j}}\,\underline{u} = 0, \quad \forall \underline{u} \in \tilde{D}(\mathrm{I}) \tag{5.12}$$

$$\underline{\underline{a}}\,\tilde{D}(\mathrm{I}) = \tilde{D}(\mathrm{I}) \quad \text{and} \quad \underline{\underline{j}}\,\tilde{D}(\mathrm{I}) = \{0\} \tag{5.13}$$

And

$$\begin{aligned}
\bar{D}(\bar{\Omega}) &= \left\{ \underline{u} \in \tilde{D}(\bar{\Omega}) \,\middle|\, \underline{\underline{j}}\,\underline{u} = 0 \right\} = \left\{ \underline{u} \in \tilde{D}(\bar{\Omega}) \,\middle|\, \underline{\underline{a}}\,\underline{u} = \underline{u} \right\} \\
\tilde{D}_{11}(\Gamma) &= \left\{ \underline{u} \in \tilde{D}(\Gamma) \,\middle|\, \underline{\underline{a}}\,\underline{u} = 0 \right\} = \left\{ \underline{u} \in \tilde{D}(\Gamma) \,\middle|\, \underline{\underline{j}}\,\underline{u} = \underline{u} \right\} \\
\tilde{D}_{12}(\Gamma) &= \left\{ \underline{u} \in \tilde{D}(\Gamma) \,\middle|\, \underline{\underline{j}}\,\underline{u} = 0 \right\} = \left\{ \underline{u} \in \tilde{D}(\Gamma) \,\middle|\, \underline{\underline{a}}\,\underline{u} = \underline{u} \right\}
\end{aligned} \tag{5.14}$$

It should also be noticed that, in view of the above relations, each $\underline{u} \in \tilde{D}(\bar{\Omega})$ can be written uniquely as

$$\underline{u} = \underline{u}_{\mathrm{I}} + \underline{u}_{\Gamma} = \underline{u}_{\mathrm{I}} + \underline{u}_{\Gamma 1} + \underline{u}_{\Gamma 2} \quad \text{with } \underline{u}_{\mathrm{I}} \in \tilde{D}(\mathrm{I}),\ \underline{u}_{\Gamma 1} \in \tilde{D}_{11}(\Gamma) \text{ and } \underline{u}_{\Gamma 2} \in \tilde{D}_{12}(\Gamma) \tag{5.15}$$

and

$$\underline{u} = \underline{u}_{\Pi} + \underline{u}_{\Delta} = \underline{u}_{\Pi} + \underline{u}_{\Delta 1} + \underline{u}_{\Delta 2} \quad \text{with } \underline{u}_{\Pi} \in \tilde{D}(\Pi),\ \underline{u}_{\Delta 1} \in \tilde{D}_{11}(\Delta) \text{ and } \underline{u}_{\Delta 2} \in \tilde{D}_{12}(\Delta) \tag{5.16}$$

VI. THE DUAL-PRIMAL SUBSPACE

For each $k \in \Omega = \{E\}$, we define the "jump-matrix at k," to be

$$\underline{\underline{j}}^{k} \equiv \left(j^{k}_{(i,\alpha)(j,\beta)} \right) \tag{6.1}$$

where:

$$j^{k}_{(i,\alpha)(j,\beta)} \equiv \left(\delta_{\alpha\beta} - \frac{1}{m(k)} \right) \delta_{ik} \delta_{jk} \tag{6.2}$$

For any vecto $\underline{\mathrm{v}} \in \tilde{D}(\bar{\Omega})$ the condition that $\underline{\underline{j}}^{k}\,\underline{\mathrm{v}} = 0$ is tantamount to

$$\underline{\mathrm{v}}(k,\alpha) = \underline{\mathrm{v}}(k,\beta), \quad \forall \alpha,\ \beta \in \mathrm{Z}(k). \tag{6.3}$$

When $\underline{\underline{j}}^{k}\,\underline{\mathrm{v}} = 0$, we say that $\underline{\mathrm{v}}$ is continuous at the node k.

The "primal jump" matrix is defined to be

$$\underline{\underline{j}}^{\pi} \equiv \sum_{k \in \Omega^{\pi}} \underline{\underline{j}}^{k} \tag{6.4}$$

Introducing the symbol δ_{ij}^{π}, defined by

$$\delta_{ij}^{\pi} \equiv \begin{cases} 1, & \text{if } i, j \in \Omega^{\pi} \\ 0, & \text{if } i \text{ or } j \notin \Omega^{\pi} \end{cases} \tag{6.5}$$

It is seen that

$$j_{(i,\alpha)(j,\beta)}^{\pi} = \left(\delta_{\alpha\beta} - \frac{1}{m(i)}\right) \delta_{ij} \delta_{ij}^{\pi}. \tag{6.6}$$

Then the "dual-primal" space, $\tilde{D}^{DP}(\bar{\Omega})$, is defined to be

$$\tilde{D}^{DP}(\bar{\Omega}) \equiv \left\{ \underline{w} \in \tilde{D}(\bar{\Omega}) \,\middle|\, \underline{\underline{j}}^{\pi} \underline{w} = 0 \right\} \subset \tilde{D}(\bar{\Omega}) \tag{6.7}$$

In particular, $\tilde{D}^{DP}(\bar{\Omega}) = \tilde{D}(\bar{\Omega})$ when $\Omega^{\pi} = \emptyset$. Observe that the projection $\underline{\underline{a}}^{\pi} : \tilde{D}(\bar{\Omega}) \to \tilde{D}^{DP}(\bar{\Omega})$ is given by

$$\underline{\underline{a}}^{\pi} \equiv \underline{\underline{I}} - \underline{\underline{j}}^{\pi} \tag{6.8}$$

Therefore,

$$a_{(i,\alpha)(j,\beta)}^{\pi} = \frac{1}{m(i)} \delta_{ij} \delta_{ij}^{\pi} + \delta_{\alpha\beta} \delta_{ij} \left(1 - \delta_{ij}^{\pi}\right) \tag{6.9}$$

In words, this equation says that $\underline{\underline{a}}^{\pi}$ equals the identity matrix at every derived node except when the node belongs to the set π of primal nodes, in which case it equals the average matrix as given by Eq. (5.5). Similarly, Eq. (6.6) says that the primal jump operator $\underline{\underline{j}}^{\pi}$ vanishes everywhere except at primal nodes, where it equals the jump operator. Therefore, the dual-primal space $\tilde{D}^{DP}(\bar{\Omega})$ is the subspace of $\tilde{D}(\bar{\Omega})$ whose elements are continuous at every *node* belonging to π. We adopt the notations

$$\tilde{D}_{11}^{DP}(\bar{\Omega}) \equiv \underline{\underline{j}} \tilde{D}^{DP}(\bar{\Omega}) \subset \tilde{D}^{DP}(\bar{\Omega}) \text{ and } \tilde{D}_{12}^{DP}(\bar{\Omega}) \equiv \underline{\underline{a}} \tilde{D}^{DP}(\bar{\Omega}) = \tilde{D}_{12}(\bar{\Omega}) \tag{6.10}$$

To prove that $\underline{\underline{j}} \tilde{D}^{DP}(\bar{\Omega}) \subset \tilde{D}^{DP}(\bar{\Omega})$, given $\underline{w} \in \tilde{D}^{DP}(\bar{\Omega})$ we compute the projection of $\underline{\underline{j}} \underline{w}$ on $\tilde{D}^{DP}(\bar{\Omega})$:

$$\underline{\underline{a}}^{\pi} \underline{\underline{j}} \underline{w} = \left(\underline{\underline{I}} - \underline{\underline{j}}^{\pi}\right) \underline{\underline{j}} \underline{w} = \underline{\underline{j}} \underline{w} - \underline{\underline{j}}^{\pi} \underline{w} = \underline{\underline{j}} \underline{w}. \tag{6.11}$$

VII. THE DISCONTINUOUS MULTIPLIERS-FREE FORMULATION

In the remaining of this article, several matrices will be considered.

$$\underline{\underline{\hat{A}}} : \tilde{D}(\Omega) \to \tilde{D}(\Omega), \quad \underline{\underline{A}}^{t} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega}) \quad \text{and} \quad \underline{\underline{A}} : \tilde{D}(\bar{\Omega}) \to \tilde{D}^{DP}(\bar{\Omega}) \tag{7.1}$$

The matrix $\underline{\underline{\hat{A}}}$, $\underline{\underline{A}}^{t}$, and $\underline{\underline{A}}$ will be referred to as the "original matrix," the "total matrix" and the "dual-primal matrix," respectively. We write:

$$\underline{\underline{\hat{A}}} \equiv \left(\widehat{A}_{pq} \right), \quad \text{where } p, q \in \Omega \tag{7.2}$$

It will be assumed throughout the article that:

1. $\underline{\underline{\hat{A}}} : \tilde{D}(\Omega) \to \tilde{D}(\Omega)$ is positive definite; and
2. Using the notation of Eq. (7.2),

$$\widehat{A}_{pq} = 0, \quad \text{whenever } p \in \Omega^{\mathrm{I}} \cap \Omega_\alpha,\ q \in \Omega^{\mathrm{I}} \cap \Omega_\beta \text{ and } \alpha \neq \beta \tag{7.3}$$

3. The matrix $\underline{\underline{A}}^t : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ is positive definite and satisfies the condition:

$$\underline{\widehat{w}} \bullet \underline{\underline{\widehat{A}}}\,\underline{\widehat{u}} = \tau\big(\underline{\widehat{w}}\big) \bullet \underline{\underline{A}}^t \tau\big(\underline{\widehat{u}}\big), \quad \forall \underline{\widehat{u}}, \underline{\widehat{w}} \in \tilde{D}(\Omega) \tag{7.4}$$

   This condition, Eq. (7.4), does not determine $\underline{\underline{A}}^t$ uniquely.
4. For each $\alpha \in \{1, \ldots, E\}$ there is defined a matrix $\underline{\underline{A}}^\alpha : \tilde{D}(\bar{\Omega}_\alpha) \to \tilde{D}(\bar{\Omega}_\alpha)$ such that

$$\underline{\underline{A}}^t = \sum_{\alpha=1}^{E} \underline{\underline{A}}^\alpha \tag{7.5}$$

   A convenient procedure for constructing a matrix $\underline{\underline{A}}^t$ fulfilling the above conditions is given in Appendix A, proving thereby that there is always at least one such a matrix.
5. When $\underline{\underline{A}}^t$ is given, the dual-primal matrix, $\underline{\underline{A}}$, is defined by

$$\underline{\underline{A}} \equiv \underline{\underline{a}}^\pi \underline{\underline{A}}^t \underline{\underline{a}}^\pi \tag{7.6}$$

6. The dual-primal subspace of $\tilde{D}(\bar{\Omega})$ is defined to be

$$\tilde{D}^{DP}(\bar{\Omega}) \equiv \underline{\underline{a}}^\pi \tilde{D}(\bar{\Omega}) \tag{7.7}$$

We observe that when $\pi \subset \bar{\Omega}$ is void, $\underline{\underline{A}} = \underline{\underline{A}}^t$ and $\tilde{D}^{DP}(\bar{\Omega}) = \tilde{D}(\bar{\Omega})$. Furthermore, the matrix $\underline{\underline{A}}$ defines a mapping $\underline{\underline{A}} : \tilde{D}^{DP}(\bar{\Omega}) \to \tilde{D}^{DP}(\bar{\Omega})$, which is symmetric and positive definite. This because

$$\underline{w} \bullet \underline{\underline{A}}\,\underline{u} = \underline{w} \bullet \underline{\underline{a}}^\pi \underline{\underline{A}}^t \underline{\underline{a}}^\pi \underline{u} = \underline{w} \bullet \underline{\underline{A}}^t \underline{u}, \quad \forall \underline{u}, \underline{w} \in \tilde{D}^{DP}(\bar{\Omega}) \tag{7.8}$$

Observe that $\underline{\underline{A}} : \tilde{D}^{DP}(\bar{\Omega}) \to \tilde{D}^{DP}(\bar{\Omega})$ is positive definite and one-to-one, while $\underline{\underline{A}} : \tilde{D}(\bar{\Omega}) \to \tilde{D}^{DP}(\bar{\Omega})$ is not. Furthermore:

$$\begin{aligned} &\tilde{D}(\bar{\Omega}) = \tilde{D}_{12}(\bar{\Omega}) = \underline{\underline{a}} \tilde{D}^{DP}(\bar{\Omega}) \subset \tilde{D}^{DP}(\bar{\Omega}) \\ &\tilde{D}(\Delta) \subset \tilde{D}^{DP}(\bar{\Omega}), \quad \tilde{D}_{11}(\Delta) \equiv \underline{\underline{j}} \tilde{D}(\Delta) = \underline{\underline{j}} \tilde{D}^{DP}(\bar{\Omega}) \end{aligned} \tag{7.9}$$

Definition 7.1. *Let $\underline{\widehat{f}} \in \tilde{D}(\Omega)$. Then the "original problem" consists in searching for a function $\underline{\widehat{u}} \in \tilde{D}(\Omega)$ that satisfies*

$$\underline{\underline{\widehat{A}}}\,\underline{\widehat{u}} = \underline{\widehat{f}} \tag{7.10}$$

The "transformed problem" consists in searching for a function $\underline{\tilde{u}} \in \tilde{D}^{DP}(\bar{\Omega})$ *that satisfies*

$$\underline{\underline{aA}}\,\underline{\tilde{u}} = \underline{\bar{f}} \quad \textit{and} \quad \underline{\underline{j}}\,\underline{\tilde{u}} = 0 \tag{7.11}$$

where $\underline{\bar{f}} \in \bar{D}(\bar{\Omega}) = \tilde{D}_{12}(\bar{\Omega}) \subset \tilde{D}^{DP}(\bar{\Omega})$ *is given by*

$$\underline{\bar{f}} \equiv \begin{pmatrix} \underline{\bar{f}}_{\Pi} \\ \underline{\bar{f}}_{\Delta} \end{pmatrix} \equiv \underline{\underline{m}}^{-1}\tau \begin{pmatrix} \widehat{\underline{f}} \end{pmatrix} \quad \textit{and} \quad \underline{\bar{f}}_{\Delta} = \underline{\bar{f}}_{\Delta 2} \tag{7.12}$$

Theorem 7.1. *An equivalent formulation of the transformed problem is: search for a function* $\underline{\tilde{u}} \in \tilde{D}(\bar{\Omega})$ *that satisfies*

$$\underline{\underline{aA}}^{t}\,\underline{\tilde{u}} = \underline{\bar{f}} \quad \textit{and} \quad \underline{\underline{j}}\,\underline{\tilde{u}} = 0 \tag{7.13}$$

Furthermore, a function $\underline{\tilde{u}} \in \tilde{D}(\bar{\Omega})$ *is the solution of the transformed problem if and only if*

$$\widehat{\underline{u}} \equiv \tau^{-1}(\underline{\tilde{u}}) \tag{7.14}$$

is the solution of the original problem.

Proof. start with, we prove that both formulations, mentioned earlier, of the transformed problem are equivalent. This can be seen using the fact that when $\underline{\underline{j}}\,\underline{\tilde{u}} = 0$, then $\underline{\tilde{u}} \in \tilde{D}_{12}(\bar{\Omega}) \subset \tilde{D}^{DP}(\bar{\Omega})$ and also

$$\underline{\underline{aA}}^{t}\,\underline{\tilde{u}} = \underline{\underline{aa}}^{\pi}\,\underline{\underline{A}}^{t}\,\underline{\underline{a}}^{\pi}\,\underline{\tilde{u}} = \underline{\underline{aA}}\,\underline{\tilde{u}} \tag{7.15}$$

Recall now the definition of the transformation $\tau^{-1} : \tilde{D}_{12}(\bar{\Omega}) \to \tilde{D}(\Omega)$ given in Section III and assume that $\underline{\tilde{u}} \in \tilde{D}(\bar{\Omega})$ is related to $\widehat{\underline{u}} \in \tilde{D}(\Omega)$ by Eq. (7.14). Then, we have

1. If $\widehat{\underline{u}} \in \tilde{D}(\Omega)$ is solution of the original problem, then $\underline{\tilde{u}} \equiv \tau\left(\widehat{\underline{u}}\right)$ fulfills Eq. (7.13);
2. Conversely, Eq. (7.13) implies $\underline{\tilde{u}} \in \bar{D}(\bar{\Omega})$, so that τ^{-1} is well defined. Taking $\widehat{\underline{u}} \in \tilde{D}(\Omega)$ given by Eq. (7.14), then following the above arguments in reverse order, it is seen that $\widehat{\underline{u}} \in \tilde{D}(\Omega)$ fulfills Eq. (7.10). ∎

VIII. GREEN–HERRERA FORMULA FOR MATRICES

In what follows we shall write

$$\underline{\underline{A}} \equiv \begin{pmatrix} \underline{\underline{A}}_{\Pi\Pi} & \underline{\underline{A}}_{\Pi\Delta} \\ \underline{\underline{A}}_{\Delta\Pi} & \underline{\underline{A}}_{\Delta\Delta} \end{pmatrix} \tag{8.1}$$

The notation here is such that

$$\begin{cases} \underline{\underline{A}}_{\Pi\Pi} : \tilde{D}(\Pi) \to \tilde{D}(\Pi), & \underline{\underline{A}}_{\Pi\Delta} : \tilde{D}(\Delta) \to \tilde{D}(\Pi) \\ \underline{\underline{A}}_{\Delta\Pi} : \tilde{D}(\Pi) \to \tilde{D}(\Delta), & \underline{\underline{A}}_{\Delta\Delta} : \tilde{D}(\Delta) \to \tilde{D}(\Delta) \end{cases} \tag{8.2}$$

And the following definitions are introduced

$$\underline{\underline{L}} \equiv \begin{pmatrix} \underline{\underline{A}}_{\Pi\Pi} & \underline{\underline{A}}_{\Pi\Delta} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \underline{\underline{R}} \equiv \begin{pmatrix} 0 & 0 \\ \underline{\underline{A}}_{\Delta\Pi} & \underline{\underline{A}}_{\Delta\Delta} \end{pmatrix} \tag{8.3}$$

Furthermore, we notice the identity:

$$\underline{\underline{R}} = \underline{\underline{a}}\,\underline{\underline{R}} + \underline{\underline{j}}\,\underline{\underline{R}} \tag{8.4}$$

Which implies, since $\underline{\underline{A}} = \underline{\underline{A}}^T$, that

$$\underline{\underline{L}} + \underline{\underline{a}}\,\underline{\underline{R}} + \underline{\underline{j}}\,\underline{\underline{R}} = \underline{\underline{L}}^T + \underline{\underline{R}}^T\,\underline{\underline{a}} + \underline{\underline{R}}^T\,\underline{\underline{j}} \tag{8.5}$$

The identity:

$$\underline{\underline{L}} + \underline{\underline{a}}\,\underline{\underline{R}} - \underline{\underline{R}}^T\,\underline{\underline{j}} = \underline{\underline{L}}^T + \underline{\underline{R}}^T\,\underline{\underline{a}} - \underline{\underline{j}}\,\underline{\underline{R}} \tag{8.6}$$

which follows from Eq. (8.5), will be referred to as "Green-Herrera formula for matrices".

We notice that the ranges of $\underline{\underline{L}}$ and $\underline{\underline{R}}$ are contained in $\tilde{D}(\Pi)$ and $\tilde{D}(\Delta)$, respectively, whereas those of $\underline{\underline{a}}\,\underline{\underline{R}}$ and $\underline{\underline{j}}\,\underline{\underline{R}}$ are contained in $\tilde{D}_{12}(\Delta)$ and $\tilde{D}_{11}(\Delta)$, respectively; so, the ranges of $\underline{\underline{L}}$, $\underline{\underline{a}}\,\underline{\underline{R}}$, and $\underline{\underline{j}}\,\underline{\underline{R}}$ are linearly independent. Furthermore, for any function $\underline{\mathrm{v}} \in \tilde{D}^{DP}(\bar{\Omega})$ one has

$$\left(\underline{\underline{L}} + \underline{\underline{a}}\,\underline{\underline{R}} - \underline{\underline{R}}^T\,\underline{\underline{j}}\right)\underline{\mathrm{v}} = 0 \tag{8.7}$$

if and only if

$$\underline{\underline{L}}\,\underline{\mathrm{v}} = 0, \quad \underline{\underline{a}}\,\underline{\underline{R}}\,\underline{\mathrm{v}} = 0 \quad \text{and} \quad \underline{\underline{j}}\,\underline{\mathrm{v}} = 0 \tag{8.8}$$

To establish the equivalence between Eqs. (8.7) and (8.8), one can use the facts that the ranges of $\underline{\underline{L}}$ and $\underline{\underline{a}}\,\underline{\underline{R}}$ are linearly independent, together with the equation:

$$\left(\underline{\underline{j}}\,\underline{\mathrm{v}}\right) \bullet \underline{\underline{R}}^T\,\underline{\underline{j}}\,\underline{\mathrm{v}} = \left(\underline{\underline{j}}\,\underline{\mathrm{v}}\right) \bullet \underline{\underline{A}}_{\Delta\Delta}\,\underline{\underline{j}}\,\underline{\mathrm{v}} = \left(\underline{\underline{j}}\,\underline{\mathrm{v}}\right) \bullet \underline{\underline{A}}\,\underline{\underline{j}}\,\underline{\mathrm{v}} = 0 \tag{8.9}$$

which implies $\underline{\underline{j}}\,\underline{\mathrm{v}} = 0$. This, because $\underline{\underline{A}}$ is positive definite on $\tilde{D}^{DP}(\bar{\Omega})$.

IX. THE STEKLOV–POINCARÉ OPERATOR

In this Section, we use the following notation:

$$\overset{\bullet}{\underline{\hat{u}}} \equiv \underline{\underline{a}}\,\underline{u} \text{ and } [\![\underline{u}]\!] \equiv \underline{\underline{j}}\,\underline{u} \tag{9.1}$$

Then $\overset{\bullet}{\underline{\hat{u}}} \in \bar{D}(\bar{\Omega})$, while $[\![\underline{u}]\!]$ belongs to $\tilde{D}_{11}(\Gamma) \subset \tilde{D}(\Gamma)$. The Green–Herrera formula of Eq. (8.6), is equivalent to:

$$\underline{w} \bullet \underline{\underline{L}}\,\underline{u} + \overset{\bullet}{\underline{\hat{w}}} \bullet \underline{\underline{a}}\,\underline{\underline{R}}\,\underline{u} - [\![\underline{u}]\!]\,\underline{\underline{j}}\,\underline{\underline{R}}\,\underline{w} = \underline{u} \bullet \underline{\underline{L}}\,\underline{w} + \overset{\bullet}{\underline{\hat{u}}} \bullet \underline{\underline{a}}\,\underline{\underline{R}}\,\underline{w} - [\![\underline{w}]\!]\,\underline{\underline{j}}\,\underline{\underline{R}}\,\underline{u}, \quad \forall \underline{u}, \underline{w} \in \tilde{D}(\bar{\Omega}) \tag{9.2}$$

Now, Green–Herrera formulas were originally introduced for partial differential operators acting on discontinuous functions [5]; they can be applied to any such an operator when it is linear. Eq. (8.6), on the other hand, is intended as an extension of such kind of formulas to matrices acting on discontinuous vectors and it has interest to compare Eq. (9.2) with Green–Herrera formulas for partial differential operators. To this end it is useful to introduce the following notation

$$\left[\!\left[\underline{\underline{R}}\right]\!\right] \equiv -\underline{\underline{aR}} \text{ and } \overset{\bullet}{\underline{\underline{\hat{R}}}} \equiv -\underline{\underline{jR}} \tag{9.3}$$

Using it, Eq. (9.2) is

$$\underline{w} \bullet \underline{\underline{L}}\underline{u} + [\![\underline{u}]\!] \bullet \overset{\bullet}{\underline{\underline{\hat{R}}}}\underline{w} - \overset{\bullet}{\underline{\hat{w}}} \bullet \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{u} = \underline{u} \bullet \underline{\underline{L}}\underline{w} + [\![\underline{w}]\!] \bullet \overset{\bullet}{\underline{\underline{\hat{R}}}}\underline{u} - \overset{\bullet}{\underline{\hat{u}}} \bullet \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{w}, \forall \underline{u}, \underline{w} \in \tilde{D}(\bar{\Omega}) \tag{9.4}$$

There is a straightforward correspondence between Eq. (9.4) and what is obtained for differential operators (see [8,9] for general results and many illustrations). For Laplace operator, for example, they are

$$\int_\Omega w\mathcal{L}u dx + \int_\Gamma \left\{ [\![u]\!] \overset{\bullet}{\widehat{\frac{\partial w}{\partial n}}} - \overset{\bullet}{\hat{w}} \left[\!\left[\frac{\partial u}{\partial n} \right]\!\right] \right\} dx = \int_\Omega u\mathcal{L}w dx + \int_\Gamma \left\{ [\![w]\!] \overset{\bullet}{\widehat{\frac{\partial u}{\partial n}}} - \overset{\bullet}{\hat{u}} \left[\!\left[\frac{\partial w}{\partial n} \right]\!\right] \right\} dx \tag{9.5}$$

The following correspondence between the bilinear functionals involved in both equations, stem by comparison of Eqs. (9.4) and (9.5):

$$\begin{aligned} &\int_\Omega w\mathcal{L}u dx \leftrightarrow \underline{w} \bullet \underline{\underline{L}}\underline{u} \\ &\int_\Gamma [\![u]\!] \overset{\bullet}{\widehat{\frac{\partial w}{\partial n}}} dx \leftrightarrow [\![\underline{u}]\!] \bullet \overset{\bullet}{\underline{\underline{\hat{R}}}}\underline{w} \\ &\int_\Gamma \overset{\bullet}{\hat{w}} \left[\!\left[\frac{\partial u}{\partial n} \right]\!\right] dx \leftrightarrow \overset{\bullet}{\underline{\hat{w}}} \bullet \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{u} \end{aligned} \tag{9.6}$$

For partial differential operators and in particular for Laplace operator, the Poincaré–Steklov operator is associated with the jump of the normal derivative and the bilinear functional

$$\int_\Gamma \overset{\bullet}{\hat{w}} \left[\!\left[\frac{\partial u}{\partial n} \right]\!\right] dx \tag{9.7}$$

Hence, at the matrix level the Steklov–Poincaré operator is associated with the bilinear from:

$$\overset{\bullet}{\underline{\hat{w}}} \bullet \left[\!\left[\underline{\underline{R}}\right]\!\right]\underline{u} = \underline{w} \bullet \underline{\underline{aR}}\underline{u}, \quad \forall \underline{u}, \ \underline{w} \in \tilde{D}(\bar{\Omega}) \tag{9.8}$$

Or more simply, with the matrix $\underline{\underline{aR}}$. Thus, we define the Steklov–Poincaré operator to be the matrix $\underline{\underline{aR}}$.

Correspondences similar to those of Eq. (9.6) can be established in general; applications include the governing system of equations of linear elasticity and many other problems [8], which cover a

great variety of scientific and engineering systems. This definition of the Steklov–Poincaré operator differs from conventional interpretations that have been presented by many authors (compare for example with [2], pp 3 and 4, or [3], pp 3, 46, and 47,), but it is more adequate in several respects; a basic advantage of the new definition is that when it is used the Steklov–Poincaré operator for matrices is a linear operator, a property that is not enjoyed by conventional definitions.

X. APPLICATION OF THE GREEN–HERRERA FORMULA TO THE TRANSFORMED PROBLEM

In [6] and [7] Green–Herrera formula was applied to the transformed problem. The next theorem contains such results.

Theorem 10.1. *Let* $\underline{\bar{f}} \in \bar{D}(\bar{\Omega}) = \tilde{D}_{12}(\bar{\Omega})$ *be defined by Eq.* (7.10)*, then a* $\underline{\tilde{\mathrm{v}}} \in \tilde{D}(\bar{\Omega})$ *satisfies*

$$\left(\underline{\underline{L}} + \underline{\underline{aR}} - \underline{\underline{R}}^T \underline{\underline{j}}\right) \underline{\tilde{\mathrm{v}}} = \underline{\bar{f}} \tag{10.1}$$

If and only if, $\underline{\tilde{\mathrm{v}}}$ *is the solution of the transformed problem.*

Proof. Let $\underline{\tilde{u}} \in \tilde{D}(\bar{\Omega})$ be the solution of the transformed problem and assume $\underline{\tilde{\mathrm{v}}} \in \tilde{D}(\bar{\Omega})$ fulfills Eq. (10.1). Then

$$\left(\underline{\underline{L}} + \underline{\underline{aR}} - \underline{\underline{R}}^T \underline{\underline{j}}\right) \underline{\tilde{u}} = \left(\underline{\underline{L}} + \underline{\underline{aR}}\right) \underline{\tilde{u}} = \underline{\underline{aA}}\,\underline{\tilde{u}} = \underline{\bar{f}} \tag{10.2}$$

To prove the converse, define $\underline{w} \equiv \underline{\tilde{\mathrm{v}}} - \underline{\tilde{u}}$. Then,

$$\left(\underline{\underline{L}} + \underline{\underline{aR}} - \underline{\underline{R}}^T \underline{\underline{j}}\right) \underline{w} = \underline{0} \tag{10.3}$$

Using the results of Section VIII, Eq. (8.8) it is seen that $\underline{\tilde{\mathrm{v}}} = \underline{\tilde{u}} + \underline{w}$ fulfills

$$\underline{\underline{A}}\,\underline{\tilde{\mathrm{v}}} = \left(\underline{\underline{L}} + \underline{\underline{aR}}\right) \underline{\tilde{\mathrm{v}}} = \left(\underline{\underline{L}} + \underline{\underline{aR}}\right) \underline{\tilde{u}} = \underline{\underline{aA}}\,\underline{\tilde{u}} = \underline{\bar{f}} \quad \text{and} \quad \underline{\underline{j}}\,\underline{\tilde{\mathrm{v}}} = \underline{\underline{j}}\,\underline{\tilde{u}} = 0 \tag{10.4}$$

■

XI. FORMULATION IN TERMS OF SCHUR COMPLEMENT MATRICES

It is advantageous to transform the problem we are considering into one in which the right-hand side of Eq. (10.1) belongs to $\tilde{D}(\Delta)$. This is achieved by subtracting the auxiliary vector

$$\underline{u}_P \equiv \underline{\underline{A}}^{-1}_{\Pi\Pi}\,\underline{\bar{f}}_{\Pi} \tag{11.1}$$

We notice that Eq. (11.1) implies

$$(\underline{u}_P)_\Delta = 0 \tag{11.2}$$

Therefore, $\underline{\underline{j}}\,\underline{u}_P = 0$. Defining $\underline{u} \equiv \underline{\tilde{u}} - \underline{u}_P$, then Eq. (10.1) becomes

$$\left(\underline{\underline{L}} + \underline{\underline{aR}} - \underline{\underline{R}}^T \underline{\underline{j}}\right) \underline{u} = \underline{\bar{f}}_{\Delta 2} - \underline{\underline{aA}}_{\Delta\Pi}\,\underline{\underline{A}}^{-1}_{\Pi\Pi}\,\underline{\bar{f}}_{\Pi} \equiv \underline{f}_{\Delta 2} \tag{11.3}$$

Here, $\underline{f}_{\Delta 2} \in \tilde{D}_{12}(\Delta)$ is defined as indicated in Eq. (11.3), which in view of Eq. (8.8) is equivalent to

$$\underline{\underline{L}}\,\underline{u} = 0, \quad \underline{\underline{aR}}\,\underline{u} = \underline{f}_{\Delta 2} \quad \text{and} \quad \underline{\underline{j}}\,\underline{u} = 0 \tag{11.4}$$

The "harmonic functions space," is defined to be

$$D \equiv \left\{ \underline{u} \in \tilde{D}(\bar{\Omega}) | L\underline{u} = 0 \right\} \tag{11.5}$$

Hence, the problem of Eq. (11.3) can be stated as: Find a harmonic vector (i.e., such that $\underline{u} \in D$) that satisfies

$$\left(\underline{\underline{aR}} - \underline{\underline{R}}^T \underline{\underline{j}} \right) \underline{u} = \underline{f}_{\Delta 2} \tag{11.6}$$

Some important properties of harmonic functions are listed next.

A. Harmonic functions are characterized by their dual-values. Indeed, if $\underline{u} \in D$, then

$$\underline{u}_\Pi = -\underline{\underline{A}}_{\Pi\Pi}^{-1} \underline{\underline{A}}_{\Pi\Delta} \underline{u}_\Delta \tag{11.7}$$

B. Every harmonic function $\underline{u} \in D$ belongs to $\tilde{D}^{DP}(\Delta)$; i.e., $D \subset \tilde{D}^{DP}(\Delta)$;
C. When $\underline{u} \in D$,

$$\underline{\underline{A}}\,\underline{u} = \underline{\underline{R}}\,\underline{u} = \underline{\underline{S}}\,\underline{u} \tag{11.8}$$

where $\underline{\underline{S}}$ is the "dual-primal Schur complement matrix," defined by

$$\underline{\underline{S}} \equiv \underline{\underline{A}}_{\Delta\Delta} - \underline{\underline{A}}_{\Delta\Pi} \underline{\underline{A}}_{\Pi\Pi}^{-1} \underline{\underline{A}}_{\Pi\Delta} \tag{11.9}$$

D. Furthermore, the matrix $\underline{\underline{S}}$ defines a transformation, $\underline{\underline{S}} : \tilde{D}(\Delta) \to \tilde{D}(\Delta)$, of $\tilde{D}(\Delta)$ into itself, which is symmetric and positive definite.

In the next theorem we show that Eq. (11.6) can be replaced by

$$\left(\underline{\underline{aS}} - \underline{\underline{S}}\,\underline{\underline{j}} \right) \underline{u}_\Delta = \underline{f}_{\Delta 2} \tag{11.10}$$

Therefore, when harmonic functions are used, our problem can be stated as follows:

"Find a harmonic function, $\underline{u} \in D$, whose dual-values satisfy Eq. (11.10)."

This formulation will be referred to as the "dual-values formulation." Furthermore, using arguments similar to those of Section VIII, it can be shown that Eq. (11.10) is equivalent to

$$\underline{\underline{aS}}\,\underline{u}_\Delta = \underline{f}_{\Delta 2} \quad \text{and} \quad \underline{\underline{j}}\,\underline{u}_\Delta = 0 \tag{11.11}$$

Theorem 11.1. *Let* $\underline{u} \equiv \underline{u}_\Pi + \underline{u}_\Delta \in D$. *Then, Eqs* (11.6), (11.10), *and* (11.11) *are equivalent.*

Proof. The equivalence between Eqs. (11.10) and (11.11) was already established; thus, it is enough to prove that Eqs (11.6) and (11.11) are equivalent. This is immediate because, when $\underline{u} \in D$, Eq. (11.6) is equivalent to the transformed problem since

$$\underline{\underline{aA}}\,\underline{u} = \underline{\underline{aS}}\,\underline{u}_\Delta \quad \text{and} \quad \underline{\underline{j}}\,\underline{u} = \underline{\underline{j}}(\underline{u}_\Pi + \underline{u}_\Delta) = \underline{\underline{j}}\,\underline{u}_\Delta \tag{11.12}$$

■

In what follows these results will be used to derive a wide variety of nonoverlapping domain decomposition methods, which permit obtaining the boundary-values, $\underline{u}_\Delta \in \tilde{D}(\Delta)$. Once $\underline{u}_\Delta$ is known, $\underline{u}_\Pi \in \tilde{D}(\Pi)$ is obtained by means of Eq. (11.7).

XII. ONE-WAY METHODS

As we have done up to now, in this Section and the following one we deal with the case when the matrix $\underline{\underline{A}} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ is positive definite, leaving for Sections XIV and XV the extension of the procedures here described to the case when that condition is not fulfilled.

When $\underline{\underline{A}}$ is positive definite, so is $\underline{\underline{S}} : \tilde{D}(\Delta) \to \tilde{D}(\Delta)$ and we define the energy inner product; $(\bullet, \bullet)$, on $\tilde{D}(\Delta)$, by:

$$(\underline{u}, \underline{w}) \equiv \underline{w} \bullet \underline{\underline{S}}\underline{u}, \quad \forall \underline{u}, \underline{w} \in \tilde{D}(\Delta) \tag{12.1}$$

Writing $\underline{u} = \underline{u}_\Pi + \underline{u}_\Delta$ and $\underline{w} = \underline{w}_\Pi + \underline{w}_\Delta$, we notice that

$$(\underline{u}_\Delta, \underline{w}_\Delta) = \underline{w} \bullet \underline{\underline{A}}\underline{u}, \forall \underline{u}, \underline{w} \in D \tag{12.2}$$

Problem 1. This problem consists in searching for a function $\underline{u}_\Delta \in \tilde{D}_{12}(\Delta)$ such that it satisfies

$$\underline{\underline{aS}}\underline{u}_\Delta = \underline{f}_{\Delta 2} \tag{12.3}$$

This formulation, which is based on Eq. (11.11), is suitable for the application of the conjugate gradient method (CGM) using the Euclidean inner product, because the matrix $\underline{\underline{aS}}$ defines a transformation $\tilde{D}_{12}(\Delta)$ into itself:

$$\underline{\underline{aS}} : \tilde{D}_{12}(\Delta) \to \tilde{D}_{12}(\Delta) \tag{12.4}$$

And furthermore, the matrix $\underline{\underline{aS}}$ is symmetric and positive definite on $\tilde{D}_{12}(\Delta)$. The DDM so obtained is essentially the well-known Schur complement method. However, our formulation allows the possibility of including primal nodes, something that is not usually considered.

A multipliers-free version of the nonpreconditioned FETI method can be derived from Eq. (11.10) if $\underline{u}_\Delta^{FT} \in D$ is defined as:

$$\underline{u}_\Delta^{FT} \equiv \underline{u}_\Delta - \underline{\underline{S}}^{-1}\underline{f}_{\Delta 2} \tag{12.5}$$

Then,

$$\left(\underline{\underline{aS}} - \underline{\underline{S}}\underline{\underline{j}}\right)\underline{u}_\Delta^{FT} = \underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta 2}, \tag{12.6}$$

which is fulfilled if and only if

$$\underline{\underline{aS}}\underline{u}_\Delta^{FT} = 0 \quad \text{and} \quad \underline{\underline{j}}\underline{u}_\Delta^{FT} = -\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta 2}. \tag{12.7}$$

We define the subspace $\tilde{D}_{22}(\Delta) \subset \tilde{D}(\Delta)$ by

$$\tilde{D}_{22}(\Delta) \equiv \left\{ \underline{w} \in \tilde{D}(\Delta) | \underline{\underline{a}}\underline{\underline{S}}\underline{w} = 0 \right\} \tag{12.8}$$

This subspace is the orthogonal complement, in $\tilde{D}(\Delta)$ and with respect to the energy inner product, of $\tilde{D}_{12}(\Delta)$.

Problem 2. This problem consists in searching for a function $\underline{u}_{\Delta}^{FT} \in \tilde{D}_{22}(\Delta)$ such that

$$\underline{\underline{j}}\underline{u}_{\Delta}^{FT} = -\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta 2} \tag{12.9}$$

Then, the Neumann or nonpreconditioned FETI formulation of the problem is obtained multiplying Eq. (12.9) by $\underline{\underline{S}}^{-1}$. It is defined to be:

Find a $\underline{u}_{\Delta}^{FT} \in \tilde{D}_{22}(\Delta)$ such that

$$\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{u}_{\Delta}^{FT} = -\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta 2} \tag{12.10}$$

An important property of the Neumann formulation of Eq. (12.10) is that the matrix $\underline{\underline{S}}^{-1}\underline{\underline{j}}$, when applied to any vector $\underline{v} \in \tilde{D}(\Delta)$, yields a vector $\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{v} \in \tilde{D}_{22}(\Delta)$, so that $\underline{\underline{S}}^{-1}\underline{\underline{j}} : \tilde{D}_{22}(\Delta) \rightarrow \tilde{D}_{22}(\Delta)$ is a transformation of $\tilde{D}_{22}(\Delta)$ into itself. Furthermore, the matrix $\underline{\underline{S}}^{-1}\underline{\underline{j}}$ is self-adjoint and positive definite on $\tilde{D}_{22}(\Delta)$, with respect to the energy inner product. Indeed:

$$\underline{\underline{S}}(\underline{\underline{S}}^{-1}\underline{\underline{j}}) = \underline{\underline{j}} \tag{12.11}$$

And the matrix $\underline{\underline{j}}$, which is symmetric, is in addition positive definite on $\tilde{D}_{22}(\Delta)$. This latter assertion can be seen as follows; when $\underline{u} \in \tilde{D}_{22}(\Delta)$

$$\underline{\underline{j}}\underline{u} = \underline{u} - \underline{\underline{a}}\underline{u} \text{ and } \underline{\underline{a}}\underline{u} \bullet \underline{\underline{S}}\underline{u} = 0 \tag{12.12}$$

So that

$$\underline{\underline{j}}\underline{u} \bullet \underline{\underline{S}}\underline{\underline{j}}\underline{u} = \underline{u} \bullet \underline{\underline{S}}\underline{u} + \underline{\underline{a}}\underline{u} \bullet \underline{\underline{\underline{S}}}\underline{\underline{a}}\underline{u} \tag{12.13}$$

Hence,

$$\underline{\underline{j}}\underline{u} = 0 \Rightarrow \underline{u} \bullet \underline{\underline{S}}\underline{u} + \underline{\underline{a}}\underline{u} \bullet \underline{\underline{\underline{S}}}\underline{\underline{a}}\underline{u} = \underline{\underline{j}}\underline{u} \bullet \underline{\underline{S}}\underline{\underline{j}}\underline{u} = 0 \Rightarrow \underline{u} \bullet \underline{\underline{S}}\underline{u} = 0 \Rightarrow \underline{u} = 0 \tag{12.14}$$

Taking this into account, it is seen that Eq. (12.10) is suitable for the application of the Conjugate Gradient Method CGM using the energy inner product. Furthermore, our formulations allow for the possibility of including primal nodes; i.e. $\pi \neq \emptyset$.

XIII. ROUND-TRIP METHODS

In this Section, we present two round-trip algorithms; namely, the Neumann–Neumann algorithm and the preconditioned FETI algorithm.

A. The Neumann–Neumann Algorithm

The Schur-complement iterative procedure is:

Find a $\underline{u} \in \tilde{D}_{12}(\Gamma)$ such that

$$\underline{\underline{aS}}^{-1}\underline{\underline{aS}}\underline{u}_{\Delta} = \underline{\underline{aS}}^{-1}\underline{f}_{\Delta 2} \tag{13.1}$$

Eq. (13.1) is equivalent to Eq. (12.3) because when this latter equation is multiplied by $\underline{\underline{aS}}^{-1}\underline{\underline{a}}$ Eq. (13.1) is obtained, and $\underline{\underline{aS}}^{-1}\underline{\underline{a}}$ is positive definite on $\tilde{D}_{12}(\Delta)$. This equation is suitable for the application of CGM using the energy inner product, because when $\underline{u} \in \tilde{D}_{12}(\Delta)$, then $\underline{\underline{aS}}^{-1}\underline{\underline{aS}}\underline{u} \in \tilde{D}_{12}(\Delta)$; so the matrix $\underline{\underline{aS}}^{-1}\underline{\underline{aS}}$ defines a transformation, $\underline{\underline{aS}}^{-1}\underline{\underline{aS}} : \tilde{D}_{12}(\Delta) \to \tilde{D}_{12}(\Delta)$, of $\tilde{D}_{12}(\Delta)$ into itself, which is self-adjoint and positive definite with respect to the energy inner product. This latter assertion can be seen by observing that the matrix $\underline{\underline{Sa}}\underline{\underline{S}}^{-1}\underline{\underline{aS}}$ is symmetric and positive definite. Notice, thereby, that Eq. (13.1) is a preconditioned version of Eq. (11.3), with $\underline{\underline{aS}}^{-1}$ as a preconditioner.

B. The Preconditioned FETI Algorithm

The preconditioned FETI procedure is:

Find a $\underline{u} \in \tilde{D}_{22}(\Delta)$ such that

$$\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{u}^{FT} = -\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta 2} \tag{13.2}$$

Eq. (13.2) is equivalent to Eq. (12.9) because when this latter equation is multiplied by $\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}$ Eq. (13.2) is obtained, and $\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}} : \tilde{D}_{22}(\Delta) \to \tilde{D}_{22}(\Delta)$ is nonsingular. As a matter of fact, this matrix defines a transformation of $\tilde{D}_{22}(\Delta)$ into itself that is self-adjoint and positive definite with respect to the energy inner product, as can be seen using the following facts: when $\underline{u} \in \tilde{D}_{22}(\Delta)$, then $\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{u} \in \tilde{D}_{22}(\Delta)$ and, furthermore, the matrix

$$\underline{\underline{S}}\left(\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\right) = \underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}} \tag{13.3}$$

is symmetric and positive definite on $\tilde{D}_{22}(\Delta)$. Therefore, Eq. (13.2 is suitable for applying the CGM using the energy inner product, and then the matrix to be iterated is $\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}} : \tilde{D}_{22}(\Delta) \to \tilde{D}_{22}(\Delta)$. When carrying out such a procedure, the following identity

$$\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}} = \underline{\underline{I}} - \underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{Sa}} \tag{13.4}$$

may be useful. Eq. (13.4) is a way of expressing the following result: When $\underline{u} \in \tilde{D}_{22}(\Delta)$, the

$$\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{u} = \underline{u} - \underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{Sa}}\underline{u} \tag{13.5}$$

which in turn follows from

$$\underline{u} = \underline{\underline{S}}^{-1}\underline{\underline{S}}\left(\underline{\underline{a}}\underline{u} + \underline{\underline{j}}\underline{u}\right) = \underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\left(\underline{\underline{a}}\underline{u} + \underline{\underline{j}}\underline{u}\right), \quad \forall \underline{u} \in \tilde{D}_{22}(\Delta) \tag{13.6}$$

A similar identity is fulfilled by the matrix $\underline{\underline{aS}}^{-1}\underline{\underline{aS}} : \tilde{D}_{12}(\Delta) \to \tilde{D}_{12}(\Delta)$; it is:

$$\underline{\underline{aS}}^{-1}\underline{\underline{aS}} = \underline{\underline{I}} - \underline{\underline{aS}}^{-1}\underline{\underline{j}}\underline{\underline{S}} \tag{13.7}$$

Eqs. (13.4) and (13.7) are relevant when applying CGM and when carrying out the condition number analysis.

XIV. THE LAPLACIAN-LIKE CASE

This Section is devoted to extend the theory of the preceding Sections to the case when $\underline{\underline{A}}$ is not positive definite and when, correspondingly, the Schur-complement matrix, $\underline{\underline{S}}$, lacks that property. The null subspace of $\underline{\underline{S}} : \tilde{D}(\Delta) \to \tilde{D}(\Delta)$ will be denoted by N_S; so $N_S \subset \tilde{D}(\Delta)$. Furthermore, $E \subset \tilde{D}(\Delta)$ will be the range of the matrix $\underline{\underline{S}} : \tilde{D}(\Delta) \to \tilde{D}(\Delta)$. Then,

$$\tilde{D}(\Delta) = N_S \oplus E \tag{14.1}$$

This follows because N_S and E are orthogonal complements with respect to the Euclidean inner product. The matrices $\underline{\underline{I}}^C_S : \tilde{D}(\Delta) \to N_S$ and $\underline{\underline{I}}^R_S : \tilde{D}(\Delta) \to E$ denote the projection-matrices, with respect to the Euclidean inner product, on N_S and E, respectively; i.e., for any $\underline{u} \in \tilde{D}(\Delta)$, $\underline{\underline{I}}^C_S\underline{u}$ and $\underline{\underline{I}}^R_S\underline{u}$ are such projections.

We start by noticing a fact that will be used in the sequel; namely,

$$N_S \cap \tilde{D}_{12}(\Delta) = \{0\} \tag{14.2}$$

This is clear, since $\underline{\underline{S}}$ is positive definite in the linear space of continuous vectors. We observe furthermore that Eq. (14.2) implies that $\underline{\underline{j}}$, which is non-negative on $\tilde{D}(\Delta)$, is positive definite on N_S.

We next introduce the following definitions and results. In them, the orthogonality relations and projections are understood to be with respect to the Euclidean inner product:

$$\tilde{D}^w_{11}(\Delta) \equiv \underline{\underline{j}}N_S \subset \underline{\underline{j}}\tilde{D}(\Delta) = \tilde{D}_{11}(\Delta) \tag{14.3}$$

The space $\tilde{D}^w_{12}(\Delta) \subset \tilde{D}(\Delta)$ is defined to be the orthogonal complement of $\tilde{D}^w_{11}(\Delta)$;
The matrices $\underline{\underline{j}}^w : \tilde{D}(\Delta) \to \tilde{D}^w_{11}(\Delta)$ and $\underline{\underline{a}}^w : \tilde{D}(\Delta) \to \tilde{D}^w_{12}(\Delta)$, are defined to be the projection-matrices on $\tilde{D}^w_{11}(\Delta)$ and $\tilde{D}^w_{12}(\Delta)$, respectively;

$$\underline{\underline{I}} = \underline{\underline{a}}^w + \underline{\underline{j}}^w \tag{14.4}$$

Every $\underline{u} \in \tilde{D}(\Delta)$ will be written as

$$\underline{u} = \underline{u}^w_{11} + \underline{u}^w_{12}, \quad \text{where } \underline{u}^w_{11} \equiv \underline{\underline{j}}^w\underline{u} \in \tilde{D}^w_{11}(\Delta) \text{ and } \underline{u}^w_{12} \equiv \underline{\underline{a}}^w\underline{u} \in \tilde{D}^w_{12}(\Delta) \tag{14.5}$$

When $\underline{u} \in N_S$, one has

$$\underline{u} = \underline{\underline{j}}\underline{u} + \underline{\underline{a}}\underline{u}, \quad \text{where } \underline{\underline{j}}\underline{u} \in \tilde{D}^w_{11}(\Delta) \text{ and } \underline{\underline{a}}\underline{u} \in \tilde{D}_{12}(\Delta) \subset \tilde{D}^w_{12}(\Delta) \tag{14.6}$$

Also

$$\underline{\underline{j}}^{w}\underline{u} = \underline{\underline{j}}\underline{u} \quad \text{and} \quad \underline{\underline{a}}^{w}\underline{u} = \underline{\underline{a}}\underline{u} \tag{14.7}$$

Therefore,

$$\tilde{D}_{11}^{w}(\Delta) + \tilde{D}_{12}(\Delta) \supset N_S \tag{14.8}$$

When $\underline{u} \in N_S$ and $\underline{\underline{j}}^{w}\underline{u} = 0$, then $\underline{u} = 0$. This because, when Eqs. (14.6) and (14.7) hold, so that $\underline{\underline{j}}^{w}\underline{u} = 0$ implies $\underline{u} \in \tilde{D}_{12}(\Delta)$; and recalling Eq. (14.2), $N_S \cap \tilde{D}_{12}(\Delta) = \{0\}$;
As a Corollary of 8 one has:

$$\tilde{D}_{12}^{w}(\Delta) \cap N_S = \{0\} \tag{14.9}$$

$\underline{\underline{S}}$ is positive definite on $\tilde{D}_{12}^{w}(\Delta)$, by virtue of Eq. (14.9).
The matrix $\underline{\underline{\underline{M}}} : \tilde{D}(\Delta) \to \tilde{D}(\Delta)$, defined by

$$\underline{\underline{\underline{M}}} \equiv \underline{\underline{S}} + \underline{\underline{j}}^{w} \tag{14.10}$$

is symmetric and positive definite on $\tilde{D}(\Delta)$.

Proof. Let $\underline{u} \in \tilde{D}(\Delta)$, then

$$\underline{u} \bullet \underline{\underline{\underline{M}}}\underline{u} = \underline{u} \bullet \underline{\underline{S}}\underline{u} + \underline{u} \bullet \underline{\underline{j}}^{w}\underline{u} \geq 0 \tag{14.11}$$

and the equal sign holds only when

$$\underline{\underline{S}}\underline{u} = 0 \quad \text{and} \quad \underline{\underline{j}}^{w}\underline{u} = 0 \tag{14.12}$$

Eq. (14.2) implies $\underline{u} = 0$, by 8. ∎

$$\underline{\underline{\underline{a}}}\underline{\underline{\underline{M}}} = \underline{\underline{\underline{a}}}\underline{\underline{S}} \tag{14.13}$$

because $\underline{\underline{j}}^{w}$ is the projection on $\tilde{D}_{11}^{w}(\Delta) \subset \tilde{D}_{11}(\Delta)$. Hence, $\underline{\underline{a}}\,\underline{\underline{j}}^{w} = 0$.
The equation

$$\left(\underline{\underline{\underline{a}}}\underline{\underline{\underline{M}}} - \underline{\underline{\underline{M}}}\,\underline{\underline{j}}\right)\underline{u} = \underline{f}_{\Delta 2} \tag{14.14}$$

is equivalent to

$$\underline{\underline{\underline{a}}}\underline{\underline{\underline{M}}}\underline{u} = \underline{f}_{\Delta 2} \quad \text{and} \quad \underline{\underline{j}}\underline{u} = 0 \tag{14.15}$$

which, when Eq. (14.13) is applied, reduces to

$$\underline{\underline{\underline{a}}}\underline{\underline{S}}\underline{u} = \underline{f}_{\Delta 2} \quad \text{and} \quad \underline{\underline{j}}\underline{u} = 0 \tag{14.16}$$

To finish this Section, we observe that the matrix $\underline{\underline{M}}$ can be thought as a generalization of the Schur-Complement matrix. Indeed $\underline{\underline{M}} = \underline{\underline{S}}$, when $N_S = \{0\}$, and the multipliers-free dual-primal methods, in their most general form, can be unified as follows:

ONE-WAY METHODS

$$\overset{\text{Schur}-\text{Complement}}{\underline{\underline{aM}}\,\underline{u}_\Delta = \underline{\underline{aS}}\,\underline{u}_\Delta = \underline{f}_{\Delta 2}} \qquad \overset{\text{FETI method}}{\underline{\underline{M}}^{-1}\,\underline{\underline{j}}\,\underline{u}_\Delta^{FT} = -\underline{\underline{M}}^{-1}\,\underline{\underline{j}}\,\underline{\underline{M}}^{-1}\,\underline{f}_{\Delta 2}}$$

ROUND-TRIP METHODS

$$\overset{\text{Neumann}-\text{Neumann}}{\underline{\underline{aM}}^{-1}\,\underline{\underline{aM}}\,\underline{u}_\Delta = \underline{\underline{aM}}^{-1}\,\underline{f}_{\Delta 2}} \qquad \overset{\text{Preconditioned}-\text{FETI}}{\underline{\underline{M}}^{-1}\,\underline{\underline{j}}\,\underline{\underline{M}}\,\underline{\underline{j}}\,\underline{u}^{FT} = -\underline{\underline{M}}^{-1}\,\underline{\underline{j}}\,\underline{\underline{M}}\,\underline{\underline{j}}\,\underline{\underline{M}}^{-1}\,\underline{f}_{\Delta 2}} \tag{14.17}$$

In particular, the use of $\underline{\underline{M}}$ is not required when applying the Schur-complement method.

As for nomenclature:

a. $\underline{\underline{j}}^w\,\underline{u}$ is the "weak jump" of the function $\underline{u} \in \tilde{D}(\Delta)$ and

b. $\tilde{D}_{12}^w(\Delta)$ is the space of weakly continuous functions.

XV. THE INVERSE OF THE MATRIX $\underline{\underline{M}}$

Some of the multipliers-free dual-primal methods, summarized in Eq. (14.17), require the inverse of the matrix $\underline{\underline{M}}$; so, in this Section we present a procedure for deriving it.

We start with an auxiliary result: The bilinear form $\underline{w} \bullet \underline{\underline{I}}_S^C\,\underline{u}$, with $\underline{u}, \underline{w} \in \tilde{D}_{11}^w(\Delta)$, is symmetric and positive definite.

Proof. The symmetry is clear. Furthermore,

$$\underline{u} \bullet \underline{\underline{I}}_S^C\,\underline{u} \geq 0 \tag{15.1}$$

since $\underline{\underline{I}}_S^C$ is a projection. Thus, we only need to prove that when $\underline{u} \in \tilde{D}_{11}^w(\Delta)$,

$$\underline{\underline{I}}_S^C\,\underline{u} = 0 \Rightarrow \underline{u} = 0 \tag{15.2}$$

Now, when $\underline{u} \in \tilde{D}_{11}^w(\Delta)$, one has

$$\underline{u} = \underline{\underline{j}}\,\underline{\mathrm{v}}, \qquad \text{for some } \underline{\mathrm{v}} \in N_S \tag{15.3}$$

The condition $\underline{\underline{I}}_S^C\,\underline{u} = 0$ implies that

$$\underline{w} \bullet \underline{\underline{j}}\,\underline{\mathrm{v}} = 0, \quad \forall \underline{w} \in N_S \tag{15.4}$$

Recall that $\underline{\underline{j}}$ is positive definite on N_S; hence, Eq. (15.4) implies that $\underline{\mathrm{v}} = 0$, which in view of Eq. (15.3) implies $\underline{u} = 0$. ∎

The above auxiliary result implies that $\underline{\underline{I}}_S^C : \tilde{D}_{11}^w(\Delta) \to N_S$ possesses an inverse to be denoted by $\underline{\underline{l}} : N_S \to \tilde{D}_{11}^w(\Delta)$. Then,

$$\underline{\underline{I}}_S^C \underline{\underline{l}} \underline{u} = \underline{u}, \quad \forall \underline{u} \in N_S \tag{15.5}$$

Another matrix to be used is $\underline{\underline{k}}^w : \tilde{D}(\Delta) \to \tilde{D}_{11}^w(\Delta)$. For every $\underline{u} \in \tilde{D}(\Delta)$, It is defined by

$$\underline{\underline{k}}^w \underline{u} \equiv \underline{\underline{l}} \underline{\underline{I}}_S^C \underline{u} \tag{15.6}$$

This defines $\underline{\underline{k}}^w \underline{u}$ uniquely, since $\underline{\underline{I}}_S^C \underline{u} \in N_S$. Furthermore, we observe that in view of Eq. (15.5), it has the following property: for every $\underline{u} \in \tilde{D}(\Delta)$ one has

$$\underline{\underline{I}}_S^C \underline{\underline{k}}^w \underline{u} = \underline{\underline{I}}_S^C \underline{\underline{l}} \underline{\underline{I}}_S^C \underline{u} = \underline{\underline{I}}_S^C \underline{u} \tag{15.7}$$

Therefore,

$$\underline{w} \bullet \underline{\underline{k}}^w \underline{u} = \underline{w} \bullet \underline{u}, \quad \forall \underline{w} \in N_S \tag{15.8}$$

To obtain $\underline{\underline{M}}^{-1} : \tilde{D}(\Delta) \to \tilde{D}(\Delta)$, given $\underline{u} \in \tilde{D}(\Delta)$, write $\underline{\mathrm{v}} = \underline{\underline{M}}^{-1} \underline{u}$. Then:

$$\left(\underline{\underline{S}} + \underline{\underline{j}}^w \right) \underline{\mathrm{v}} = \underline{u} \tag{15.9}$$

Applying $\underline{\underline{I}}_S^C$ to this equation, and using Eq. (15.7), one gets:

$$\underline{\underline{I}}_S^C \underline{\underline{j}}^w \underline{\mathrm{v}} = \underline{\underline{I}}_S^C \underline{u} = \underline{\underline{I}}_S^C \underline{\underline{k}}^w \underline{u} \tag{15.10}$$

We observe that $\underline{\underline{j}}^w \underline{\mathrm{v}} \in \tilde{D}_{11}^w(\Delta)$ and $\underline{\underline{k}}^w \underline{u} \in \tilde{D}_{11}^w(\Delta)$. Thus, from Eq. (15.10) it follows that

$$\underline{\underline{j}}^w \underline{\mathrm{v}} = \underline{\underline{k}}^w \underline{u} \tag{15.11}$$

Since $\underline{\underline{I}}_S^C : \tilde{D}_{11}^w(\Delta) \to N_S$ is one-to-one. By substitution of this result, Eq. (15.9) becomes

$$\underline{\underline{S}} \underline{\mathrm{v}} + \underline{\underline{k}}^w \underline{u} = \underline{u} \quad \text{or} \quad \underline{\underline{S}} \underline{\mathrm{v}} = \underline{u} - \underline{\underline{k}}^w \underline{u} \tag{15.12}$$

Now, define

$$\underline{\underline{I}}_S^R \equiv \underline{\underline{I}} - \underline{\underline{I}}_S^C, \quad \tilde{D}^R(\Delta) \equiv \underline{\underline{I}}_S^R \tilde{D}(\Delta), \quad \underline{\mathrm{v}}^R \equiv \underline{\underline{I}}_S^R \underline{\mathrm{v}} \quad \text{and} \quad \underline{\mathrm{v}}^C \equiv \underline{\underline{I}}_S^C \underline{\mathrm{v}} \tag{15.13}$$

Then, $\underline{\mathrm{v}} = \underline{\mathrm{v}}^R + \underline{\mathrm{v}}^C$, with $\underline{\mathrm{v}}^R \in \tilde{D}^R(\Delta)$ and $\underline{\mathrm{v}}^C \in N_S$. Application of $\underline{\underline{I}}_S^R$ to Eq. (15.12) yields

$$\underline{\underline{S}} \underline{\mathrm{v}}^R = \underline{\underline{I}}_S^R \left(\underline{u} - \underline{\underline{k}}^w \underline{u} \right) \tag{15.14}$$

Furthermore, $\underline{\underline{S}} : \tilde{D}^R(\Delta) \to \tilde{D}^R(\Delta)$ is one-to-one. Indeed, $\underline{\mathrm{v}}^R \in \tilde{D}^R(\Delta)$ is the unique solution of

$$\left(\underline{\underline{S}} + \underline{\underline{I}}_S^C \right) \underline{\mathrm{v}}^R = \underline{\underline{I}}_S^R \left(\underline{u} - \underline{\underline{k}}^w \underline{u} \right) \tag{15.15}$$

since the matrix $\underline{\underline{S}} + \underline{\underline{I}}_S^C$ is nonsingular. In summary:

$$\underline{v}^R = \left(\underline{\underline{S}} + \underline{\underline{I}}_S^C\right)^{-1} \underline{\underline{I}}_S^R \left(\underline{u} - \underline{\underline{k}}^w \underline{u}\right) \tag{15.16}$$

Once $\underline{v}^R \in \tilde{D}^R(\Delta)$ is available, Eq. (15.11) can be used to obtain

$$\underline{v}^C = \left(\underline{\underline{j}}^w\right)^{-1} \left(\underline{\underline{k}}^w \underline{u} - \underline{\underline{j}}^w \underline{v}^R\right) \tag{15.17}$$

Here, $(\underline{\underline{j}}^w)^{-1} : \tilde{D}_{11}^w(\Delta) \to N_S$ is the inverse of $\underline{\underline{j}}^w : N_S \to \tilde{D}_{11}^w(\Delta)$.

A last observation is that the Schur complement of the matrix

$$\begin{pmatrix} \underline{\underline{A}}_{\Pi\Pi} & \underline{\underline{A}}_{\Pi\Delta} \\ \underline{\underline{A}}_{\Delta\Pi} & \underline{\underline{A}}_{\Delta\Delta} + \underline{\underline{I}}_S^C \end{pmatrix} \tag{15.18}$$

is

$$\underline{\underline{S}} + \underline{\underline{I}}_S^C = \underline{\underline{I}}_S^C + \underline{\underline{A}}_{\Delta\Delta} - \underline{\underline{A}}_{\Delta\Pi} \underline{\underline{A}}_{\Pi\Pi}^{-1} \underline{\underline{A}}_{\Pi\Delta} \tag{15.19}$$

This may be may be used when applying $(\underline{\underline{S}} + \underline{\underline{I}}_S^C)^{-1}$. When $\pi = \emptyset$, the action of $(\underline{\underline{A}} + \underline{\underline{I}}_S^C)^{-1}$ on any vector can be computed by solving local problems exclusively.

XVI. IMPLEMENTATION ISSUES AND NUMERICAL RESULTS

The problems that were implemented are of the general elliptic form

$$\mathcal{L}u = -\nabla \cdot \left(\underline{\underline{a}} \nabla u\right) + cu = f \tag{16.1}$$

With homogeneous Dirichlet boundary conditions. In particular, the Poisson equation was included ($\underline{\underline{a}} = \underline{\underline{I}}$ and $c = 0$), which corresponds to the Laplace differential operator. The problem-domain was taken to be $\overleftrightarrow{\Omega} \subset \mathbb{R}^n$, with $n = 2, 3$. Although the algorithms were all tested with $n = 3$, the results shown in this article are for the case $n = 2$; the 3D results will be presented in a separate article, where the attention focus will be the computational efficiency and extensive comparisons with more standard approaches will be carried out.

The family of subdomains $\{\overleftrightarrow{\Omega}_1, \ldots, \overleftrightarrow{\Omega}_E\}$ is assumed to be a partition of $\overleftrightarrow{\Omega}$. Each one of such subdomains is in turn discretized by the *FEM*, using a linear basis, $\{\phi_1, \ldots, \phi_d\}$ on the entire $\overleftrightarrow{\Omega}$. On the other hand, the set $\Omega \equiv \{1, \ldots, d\}$ will be used to identify the nodes associated with $\{\phi_1, \ldots, \phi_d\}$, and the family $\{\Omega_1, \ldots, \Omega_E\}$ of subsets of Ω is defined by the condition that $p \in \Omega_\alpha$ if and only if the node associated with p belongs to the closure of $\overleftrightarrow{\Omega}_\alpha$.

Using these definitions together with those given in what follows, the multipliers-free dual-primal method presented in this article was applied. The original problem is then to solve:

$$\widehat{A}\, u = \widehat{f} \tag{16.2}$$

where

$$\widehat{A}_{ij} = \int_{\Omega} \left(\nabla \phi_i \cdot \underline{\underline{a}} \cdot \nabla \phi_j + c \phi_i \phi_j \right) dx \quad \text{and} \quad \widehat{f}_i = \int_{\Omega} f \phi_i dx \tag{16.3}$$

We define the matrix $\underline{\underline{A}}^{\alpha} : \tilde{D}(\Omega_{\alpha}) \rightarrow \tilde{D}(\Omega_{\alpha})$, for each $\alpha \in \{1, \ldots, E\}$, by:

$$A^{\alpha}_{ij} = \int_{\Omega_{\alpha}} \left(\nabla \phi_i \cdot \underline{\underline{a}} \cdot \nabla \phi_j + c \phi_i \phi_j \right) dx, \quad i, j \in \Omega_{\alpha} \tag{16.4}$$

The matrix $\underline{\underline{A}}^{t} : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}(\bar{\Omega})$ is:

$$\underline{\underline{A}}^{t} \equiv \sum_{\alpha=1}^{E} \underline{\underline{A}}^{\alpha} \tag{16.5}$$

Then, the assumptions of Section VII are fulfilled.

All the algorithms summarized in Eq. (14.17) were implemented using the CGM in the manner indicated in Appendix B. An important result is that their implementations are direct and straightforward; thus, for example 2D computational codes were easily transformed into 3D codes. What is required is the applications of the matrices $\underline{\underline{a}}$, $\underline{\underline{j}}$, $\underline{\underline{M}}$, and $\underline{\underline{M}}^{-1}$, to different vectors and all them can be computed in parallel. For the matrices $\underline{\underline{a}}$ and $\underline{\underline{j}}$ this is clear by virtue of Eqs. (5.5) and (5.6), while for the other two matrices, it is explained next.

We start with the case when $\underline{\underline{A}}^{t}$ is nonsingular (i.e., $c > 0$) and $\underline{\underline{M}} = \underline{\underline{S}}$. Then, two different situations can be distinguished. First, when no primal nodes are incorporated the parallelization of the algorithm stems from the fact that $\underline{\underline{A}}^{-1} = (\underline{\underline{A}}^{t})^{-1} = \sum_{\alpha=1}^{E} (\underline{\underline{A}}^{\alpha})^{-1}$, and the application of each $(\underline{\underline{A}}^{\alpha})^{-1}$ gives rise to a local problem whose parallel processing is straightforward. Second, when primal nodes are incorporated the parallel processing can be carried out as it is indicated in Appendix C. On the other hand, when $\underline{\underline{A}}^{t}$ is singular the procedures of Sections XIV and XV, together with those of Appendix D, lead to an algorithm in which the difficulties for its fully parallel processing have been minimized. These difficulties are primarily associated with the matrix $(\underline{\underline{j}}\underline{w}^{\alpha} \bullet \underline{\underline{j}}\underline{w}^{\beta})$ of Eq. (D8), which couples several subdomains; however, the dimension of such a matrix is small compared with the number of degrees of freedom of the whole problem.

As for the numerical experiments presented in this Section, in all cases we considered homogeneous Dirichlet boundary conditions, $u = 0$ on $\partial\Omega$, where Ω is the unit square. Then, each one of our model problems was discretized using a uniform mesh, after which the domain decomposition procedure was introduced and the number of subdomains was successively increased. The numerical results are summarized in Table I. The effective condition number, "r," has been estimated for each one of these numerical experiments. For the *roundtrip algorithms* "r" is very close to $\lambda_{\max}$ because $\lambda_{\min} \approx 1$. The best performance is exhibited by the Schur roundtrip, when it is applied to the Laplace operator (Poisson equation), which yields $r \leq 1.98863$. This value is competitive with the results obtained when some of the most efficient domain decomposition methods available at present are applied to this kind of problems (see, for example, [2]). For the other model problems, the roundtrip algorithms yield $r \leq 8.54265$ and $r \leq 18.7389$, corresponding to the Schur dual-primal and FETI dual primal, respectively. As for the one-way algorithms, the most interesting numerical result correspond to the one-way dual-primal FETI, which converges in 52 iterations. This quadruples the number of iterations required by the FETI roundtrip. However, the computational effort needed in each iteration is much smaller than that of the FETI roundtrip and, therefore, in some problems it could be competitive with it.

TABLE I. Numerical results.

					PETI-DP	SCHUR DP	SCHUR DP	SCHUR	SCHUR	FETI
N	M	#Subdomains	#DoF	#Primal	Round-trip #Iterations	Round-trip			Laplace	One way
2	30	4	3,481	1	5	5	12	14	5	12
3	30	9	7,921	58	8	8	18	23	5	23
4	30	16	14,161	87	10	10	25	31	6	29
5	30	25	22,201	116	12	12	30	40	8	38
6	30	36	32,041	145	13	13	35	47	10	40
7	30	49	43,681	175	13	13	39	54	11	45
8	30	64	57,121	203	13	14	45	62	11	49
9	30	81	72,361	232	13	13	50	69	11	50
10	30	100	89,401	261	13	13	55	77	11	51
11	30	121	108,241	290	13	13	59	84	11	52
12	30	144	128,881	319	13	13	64	92	11	53
13	30	169	151,321	348	13	13	69	100	11	54
14	30	196	175,561	377	13	13	73	107	10	53
15	30	225	201,601	406	13	13	78	114	10	53
16	30	256	229,441	435	13	13	82	123	10	54
17	30	289	259,081	464	13	13	87	131	10	54
18	30	324	290,521	493	13	13	92	139	10	54
19	30	361	323,761	522	13	13	97	147	10	54
20	30	400	358,801	551	13	13	101	154	10	52
21	30	441	395,641	580	13	12	108	162	10	52
22	30	484	434,281	609	13	12	111	169	10	54
23	30	529	474,721	638	13	12	118	176	10	52
24	30	576	516,961	667	13	12	119	185	10	52
25	30	625	561,001	696	13	12	124	192	10	52
26	30	676	606,841	725	13	12	129	199	10	52
27	30	729	654,481	754	13	12	135	207	10	52
28	30	784	703,921	783	13	12	140	215	10	52
29	30	841	755,161	812	13	12	143	223	10	52
30	30	900	808,201	841	13	12	148	230	10	52

XVII. CONCLUSIONS

It has been shown that dual-primal DDMs can be formulated without recourse to Lagrange multipliers and a multipliers-free formulation of such methods has been developed. This approach yields simple unified matrix-expressions, in terms of a generalized Schur-complement matrix, which are given in Eq. (14.17). Such formulas include the most important dual-primal methods that exist at present, here grouped into two broad classes: one-way and round-trip methods. The new approach also permits treating effectively floating subdomains. As said in the Introduction, a significant simplification in the code-development of the algorithm implementations is achieved in this manner. A conspicuous feature, which contributes to the robustness exhibited by such codes, is that the derivation of the matrix-formulas is independent of the partial differential equations that originate them and of the number of dimensions of the problem; in particular, $2D$ codes can be easily transformed into $3D$ codes. In the theory here developed, two matrices were introduced -the average and jump matrices-, which by themselves represent a significant contribution because they possess superior algebraic and computational properties. In particular, the jump operator is the optimal choice for the $\boldsymbol{B}$ operator of the FETI methods [2], as was explained in [7]. These matrices correspond to generalizations of the "average" and "jump" of a function, which

can be effectively applied at the discrete level (i.e., to vectors), not only at internal-boundary-nodes but also at edges and corners. As it is well-known, the parallel implementation of dual-primal methods is impaired by the introduction of primal-nodes. When the multipliers-free formulation developed in this series of articles is applied, such a handicap is expli-citly expressed and procedures for reducing it to a minimum are given in Appendix XVII of this article. The concept of Steklov–Poincaré operator for matrices was also revised and a new version of it, which has clear advantages over standard definitions, was given. The research on which this article is based includes also extensive numerical experiments that confirm the efficiency of the multipliers-free dual-primal methods and they are here reported.

Finally, it should be mentioned that all these results stem from a "general theory of partial differential operators acting on discontinuous functions, and of matrices acting on discontinuous vectors," which has been developed through a long time span [5–7] (see [5], for references corresponding to previous years). Contrary to standard formulations in which discontinuous functions are treated as an anomaly that requires remediation by means of Lagrange multipliers, the analysis in this theory is carried out directly on function and vector spaces of discontinuous functions and vectors.

A APPENDIX 1: CONSTRUCTION OF THE MATRIX $\underline{\underline{A}}$

This Appendix is devoted to give a procedure for constructing a matrix $\underline{\underline{A}}^t : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ that satisfies the condition of Eq. (7.4) and we start by giving this condition a more explicit form.

Recall that d is the cardinality of Ω. Then observe that the set of vectors $\{\underline{e}_1, \ldots, \underline{e}_d\} \subset \tilde{D}(\Omega)$ is a basis of $\tilde{D}(\Omega)$, when for each $i \in \Omega$, $\underline{e}_i$ is defined by

$$\underline{e}_i \equiv (\delta_{i1}, \ldots, \delta_{id}) \tag{A1}$$

Here, as in what follows δ_{ij} is the Kronecker delta. The natural immersion of this set is $\{\tau(\underline{e}_1), \ldots, \tau(\underline{e}_d)\} \subset \tilde{D}(\bar{\Omega})$, where

$$\tau(\underline{e}_i)_{(j,\alpha)} = \delta_{ij}, \quad \forall (j,\alpha) \in \bar{\Omega} \tag{A2}$$

When Eq. (7.4) is applied to this latter set of vectors, a condition equivalent to it is obtained; it is:

$$\sum_{\alpha \in Z(i)} \sum_{\beta \in Z(j)} A^t_{(i,\alpha)(j,\beta)} = \widehat{A}_{ij}, \quad \forall i, j \in \Omega \tag{A3}$$

Here, for matrices such as $\underline{\underline{A}}^t : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$, we use the following notation:

$$\underline{\underline{A}}^t \equiv \left(A^t_{(i,\alpha)(j,\beta)}\right), \text{ with } (i,\alpha), (j,\beta) \in \bar{\Omega} \tag{A4}$$

Then, for each $\alpha \in \{1, \ldots, E\}$ and each pair $i, j \in \Omega$, we define the symbol δ^α_{ij}, by

$$\begin{aligned} \delta^\alpha_{ij} &\equiv 1, \quad \text{if } i, j \in \Omega_\alpha \\ \delta^\alpha_{ij} &\equiv 0, \quad \text{if } i \text{ or } j \notin \Omega_\alpha \end{aligned} \tag{A5}$$

Furthermore, the "multiplicity $m(i,j)$ of the pair (i,j)," is defined to be

$$m(i,j) \equiv \sum_{\alpha=1}^{E} \delta_{ij}^{\alpha} \tag{A6}$$

Using this notation, the condition of Eq. (7.3) is:

$$m(i,j) = 0 \Rightarrow \widehat{A}_{ij} = 0 \tag{A7}$$

The total matrix $\underline{\underline{A}}^t : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ is now defined by

$$\left.\begin{aligned} A^t_{(i,\alpha)(j,\beta)} &\equiv 0, && \text{if } m(i,j) = 0 \\ A^t_{(i,\alpha)(j,\beta)} &\equiv \tfrac{1}{m(i,j)}\ \widehat{A}_{ij}\ \delta_{ij}^{\alpha}\delta_{\alpha\beta}, && \text{if } m(i,j) \neq 0 \end{aligned}\right\} \forall (i,\alpha),(j,\beta) \in \bar{\Omega} \tag{A8}$$

It is straightforward to verify that $\underline{\underline{A}}^t$, so defined, satisfies the condition of Eq. (A3). Indeed, using Eq. (A6) one has:

$$\sum_{\alpha\in Z(i)} \sum_{\beta\in Z(j)} A^t_{(i,\alpha)(j,\beta)} = \frac{1}{m(i,j)}\ \widehat{A}_{ij} \sum_{\alpha=1}^{E}\sum_{\beta=1}^{E} \delta_{ij}^{\alpha}\delta_{\alpha\beta} = \frac{1}{m(i,j)}\ \widehat{A}_{ij} \sum_{\alpha=1}^{E} \delta_{ij}^{\alpha} = \widehat{A}_{ij} \tag{A9}$$

The matrix $\underline{\underline{A}}^t$ just defined also satisfies the Condition 4 of Section VII. Indeed, for each $\gamma = 1,\ldots,E$, define $\underline{\underline{A}}^{\gamma} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ by

$$\begin{cases} (A^{\gamma})_{(i,\alpha)(j,\beta)} \equiv \tfrac{1}{m(i,j)}\ \widehat{A}_{ij}\ \delta_{ij}^{\gamma}\delta_{\gamma\beta}\delta_{\gamma\alpha}, & \text{if } m(i,j) \neq 0 \\ (A^{\gamma})_{(i,\alpha)(j,\beta)} \equiv \widehat{A}_{ij} = 0, & \text{if } m(i,j) = 0 \end{cases} \tag{A10}$$

Then

$$\underline{\underline{A}}^t = \sum_{\gamma=1}^{E} \underline{\underline{A}}^{\gamma} \tag{A11}$$

The proof of this latter relation follows. When $m(i,j) \neq 0$, one has

$$\sum_{\gamma=1}^{E} (A^{\gamma})_{(i,\alpha)(j,\beta)} = \frac{1}{m(i,j)}\ \widehat{A}_{ij} \sum_{\gamma=1}^{E} \delta_{ij}^{\gamma}\delta_{\gamma\beta}\delta_{\gamma\alpha} = \frac{1}{m(i,j)}\ \widehat{A}_{ij}\ \delta_{ij}^{\alpha}\delta_{\alpha\beta} = A^t_{(i,\alpha)(j,\beta)} \tag{A12}$$

And when $m(i,j) = 0$:

$$\sum_{\gamma=1}^{E} (A^{\gamma})_{(i,\alpha)(j,\beta)} = 0 = A^t_{(i,\alpha)(j,\beta)} \tag{A13}$$

B APPENDIX 2: CGM IMPLEMENTATION OF THE *ROUND-TRIP* ALGORITHMS

The CGM version to be applied in what follows is given next.

"Let u^0 be given (or $u^0 = 0$) and set $r^0 = b - Au^0$, $p^0 = r^0$. For $n = 0, 1, \ldots$ let:

$$\begin{aligned}
\alpha^n &= \frac{(p^n, p^n)}{(p^n, Ap^n)} \\
u^{n+1} &= u^n + \alpha^n p^n \\
r^{n+1} &= r^n - \alpha^n A p^n \\
\beta^n &= \frac{(r^{n+1}, r^{n+1})}{(r^n, r^n)} \\
p^{n+1} &= r^{n+1} + \beta^n p^n \\
&Go\ to\ 1''
\end{aligned} \tag{B1}$$

Here, $(\bullet, \bullet)$ is the energy inner product.

A. Neumann–Neumann: According to Eq. (13.1), as well as (14.17), the equation to be solved is:

$$\underline{\underline{aS}}^{-1}\underline{\underline{aS}}\underline{u}_{\Delta} = \underline{\underline{aS}}^{-1}\underline{f}_{\Delta 2} = \underline{\underline{aS}}^{-1}\left(\underline{\bar{f}}_{\Delta 2} - \underline{\underline{aA}}_{\Delta\Pi}\underline{\underline{A}}^{-1}_{\Pi\Pi}\underline{\bar{f}}_{\Pi}\right) \tag{B2}$$

We observe that

$$\underline{\underline{aS}}^{-1}\underline{f}_{\Delta 2} = \underline{\underline{aS}}^{-1}\left(\underline{\bar{f}}_{\Delta 2} - \underline{\underline{aA}}_{\Delta\Pi}\underline{\underline{A}}^{-1}_{\Pi\Pi}\underline{\bar{f}}_{\Pi}\right) \in \tilde{D}_{12}(\Delta) \tag{B3}$$

The algorithm is as follows:
Let

$$\underline{u}^0 = 0 \tag{B4}$$

and set

$$\underline{p}^0 = \underline{r}^0 = \underline{\underline{aS}}^{-1}\underline{f}_{\Delta 2} = \underline{\underline{aS}}^{-1}\left(\underline{\bar{f}}_{\Delta 2} - \underline{\underline{aA}}_{\Delta\Pi}\underline{\underline{A}}^{-1}_{\Pi\Pi}\underline{\bar{f}}_{\Pi}\right) \tag{B5}$$

Then, for $n = 0, 1, \ldots$, do:

$$\begin{aligned}
\alpha^n &= \frac{\underline{p}^n \bullet \underline{\underline{S}}\underline{p}^n}{\underline{p}^n \bullet \underline{\underline{Sa}}\underline{\underline{S}}^{-1}\underline{\underline{aS}}\underline{p}^n} \\
\underline{u}^{n+1} &= \underline{u}^n + \alpha^n \underline{p}^n \\
\underline{r}^{n+1} &= \underline{r}^n - \alpha^n \underline{\underline{aS}}^{-1}\underline{\underline{aS}}\underline{p}^n \\
\beta^n &= \frac{\underline{r}^{n+1} \bullet \underline{\underline{S}}\underline{r}^{n+1}}{\underline{r}^n \bullet \underline{\underline{S}}\underline{r}^n} \\
\underline{p}^{n+1} &= \underline{r}^{n+1} + \beta^n \underline{p}^n \\
&Go\ to\ 1
\end{aligned} \tag{B6}$$

B. Preconditioned FETI CGM Implementation: According to Eq. (13.2), as well as (14.17), the equation to be solved is:

$$\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{u}^{FT} = -\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta 2} = -\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\left(\underline{\bar{f}}_{\Delta 2} - \underline{\underline{aA}}_{\Delta\Pi}\underline{\underline{A}}^{-1}_{\Pi\Pi}\underline{\bar{f}}_{\Pi}\right) \tag{B7}$$

We observe that

$$-\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta 2} = -\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\left(\underline{\bar{f}}_{\Delta 2} - \underline{\underline{aA}}_{\Delta\Pi}\underline{\underline{A}}^{-1}_{\Pi\Pi}\underline{\bar{f}}_{\Pi}\right) \in \tilde{D}_{22}(\Delta) \tag{B8}$$

The algorithm is as follows:
Let $\underline{u}^0 = 0$ and set

$$\underline{p}^0 = \underline{r}^0 = -\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta 2} = -\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\left(\underline{\bar{f}}_{\Delta 2} - \underline{\underline{aA}}_{\Delta\Pi}\underline{\underline{A}}^{-1}_{\Pi\Pi}\underline{\bar{f}}_{\Pi}\right)$$

Then, for $n = 0, 1, \ldots$, do:

$$\begin{aligned}
\alpha^n &= \frac{\underline{p}^n \bullet \underline{\underline{S}}\underline{p}^n}{\underline{p}^n \bullet \underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{p}^n} \\
\underline{u}^{n+1} &= \underline{u}^n + \alpha^n \underline{p}^n \\
\underline{r}^{n+1} &= \underline{r}^n - \alpha^n \underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{p}^n \\
\beta^n &= \frac{\underline{r}^{n+1} \bullet \underline{\underline{S}}\underline{r}^{n+1}}{\underline{r}^n \bullet \underline{\underline{S}}\underline{r}^n} \\
\underline{p}^{n+1} &= \underline{r}^{n+1} + \beta^n \underline{p}^n \\
&\textit{Go to } 1
\end{aligned} \tag{B9}$$

As a last remark of this Appendix, it should be noticed that in the execution of these algorithms, most of the work goes into the computation of $\underline{\underline{S}}\underline{v}$ and $\underline{\underline{S}}^{-1}\underline{v}$ for certain vectors $\underline{v}$. In the Neumann–Neumann case, each iteration of the CGM method requires two applications of $\underline{\underline{S}}$ (namely $\underline{\underline{S}}\underline{p}^n$ and $\underline{\underline{S}}\underline{r}^{n+1}$) and one application of $\underline{\underline{S}}^{-1}$ (i.e., $\underline{\underline{S}}^{-1}(\underline{\underline{aS}}\underline{p}^n)$). The preconditioned FETI case, on the other hand, requires three applications of $\underline{\underline{S}}$ (namely, $\underline{\underline{S}}\underline{p}^n$, $\underline{\underline{S}}\underline{\underline{j}}\underline{p}^n$, and $\underline{\underline{S}}\underline{r}^{n+1}$) and one application of $\underline{\underline{S}}^{-1}$ (i.e., $\underline{\underline{S}}^{-1}(\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{p}^n)$). Consequently, the Neumann–Neumann is somewhat more efficient in this respect. More thorough comparisons of the computational efficiency of the algorithms here presented, among themselves and with other well-established procedures, are underway and will appear in a forthcoming article.

C APPENDIX 3: PARALLEL COMPUTATION OF $\underline{\underline{A}}^{-1}_{\Pi\Pi}$ AND $\underline{\underline{S}}^{-1}$

This Appendix is devoted to discuss procedures that can be applied to compute in parallel the action, on any vector, of the matrices $\underline{\underline{A}}^{-1}_{\Pi\Pi}$ and $\underline{\underline{S}}^{-1}$. We will show that such a computation yields a problem, which is similar but much smaller than the original problem and, therefore, any of the multipliers-free methods described in this article is suitable for treating it.

Given $\underline{w} \in \tilde{D}^{DP}(\bar{\Omega})$, let $\underline{v} \in \tilde{D}^{DP}(\bar{\Omega})$ be such that $\underline{v} \equiv \underline{\underline{A}}^{-1}_{\Pi\Pi}\underline{w}$. Then

$$\underline{\underline{A}}_{\Pi\Pi}\underline{v} \equiv \begin{pmatrix} \underline{\underline{A}}_{\mathrm{II}} & \underline{\underline{A}}_{\mathrm{I}\pi} \\ \underline{\underline{A}}_{\pi\mathrm{I}} & \underline{\underline{A}}_{\pi\pi} \end{pmatrix} \begin{pmatrix} \underline{v}_{\mathrm{I}} \\ \underline{v}_{\pi} \end{pmatrix} = \begin{pmatrix} \underline{w}_{\mathrm{I}} \\ \underline{w}_{\pi} \end{pmatrix} \tag{C1}$$

We define the matrix $\underline{\underline{S}}^{\Pi} : \tilde{D}(\pi) \to \tilde{D}(\pi)$ by

$$\underline{\underline{S}}^{\Pi} \equiv \underline{\underline{A}}^{t}_{\pi\pi} - \underline{\underline{A}}^{t}_{\pi\mathrm{I}}\underline{\underline{A}}^{-1}_{\mathrm{II}}\underline{\underline{A}}^{t}_{\mathrm{I}\pi} \tag{C2}$$

Then, $\underline{v}_{\pi} \in \tilde{D}(\pi)$ is the solution of

$$\underline{\underline{a}}^{\pi}\,\underline{\underline{S}}^{\Pi}\underline{v}_{\pi} = \underline{w}_{\pi} - \underline{\underline{A}}_{\pi\mathrm{I}}\underline{\underline{A}}^{-1}_{\mathrm{II}}\underline{w}_{\mathrm{I}} \text{ together with } \underline{\underline{j}}^{\pi}\underline{v}_{\pi} = 0 \tag{C3}$$

While

$$\underline{v}_{\mathrm{I}} = \underline{\underline{A}}^{-1}_{\mathrm{II}}\left(\underline{w}_{\mathrm{I}} - \underline{\underline{A}}_{\mathrm{I}\pi}\underline{v}_{\pi}\right) \tag{C4}$$

Observe that when Eq. (C3) is solved iteratively, then in the process only local problems have to be solved.

As for $\underline{\underline{S}}^{-1} : \tilde{D}(\Delta) \to \tilde{D}(\Delta)$, given $\underline{w}_{\Delta} \in \tilde{D}(\Delta)$ let $\underline{v}_{\Delta} \in \tilde{D}(\Delta)$ be such that $\underline{v}_{\Delta} \equiv \underline{\underline{S}}^{-1}\underline{w}_{\Delta}$. Then

$$\underline{\underline{a}}^{\pi}\,\underline{\underline{S}}\underline{v}_{\Delta} = \underline{w}_{\Delta}, \text{ together with } \underline{\underline{j}}^{\pi}\underline{v}_{\Delta} = 0 \tag{C5}$$

This can also be written as

$$\underline{\underline{A}}\underline{v} = \underline{\underline{a}}^{\pi}\,\underline{\underline{A}}^{t}\underline{v} = \underline{w}_{\Delta}, \text{ together with } \underline{\underline{j}}^{\pi}\underline{v}_{\Delta} = 0 \tag{C6}$$

Here, $\underline{v} = \underline{v}_{\Delta} + \underline{v}_{\pi} + \underline{v}_{\mathrm{I}} \in D$ is the harmonic extension of $\underline{v}_{\Delta}$. This harmonic extension $\underline{v} \in D \subset \tilde{D}(\bar{\Omega})$ is uniquely determined by Eq. (3.6), and so is $\underline{v}_{\Delta} \in D(\Delta)$, which is equivalent to

$$\underline{\underline{a}}^{\pi}\,\underline{\underline{S}}^{t}\underline{v} = \underline{w}_{\Delta}, \text{ together with } \underline{\underline{j}}^{\pi}\underline{v}_{\Delta} = 0 \tag{C7}$$

Furthermore, we observe that Eq. (C7) can be treated iteratively, solving only local problems, by any of the multipliers-free methods introduced in this article. However, due to the small number of degrees of freedom involved the Schur-complement method is usually satisfactory.

D APPENDIX 4: COMPUTATION OF $\underline{\underline{I}}^{C}_{S}$, $\underline{\underline{j}}^{W}$, $(\underline{\underline{j}}^{W})^{-1}$, AND $\underline{\underline{k}}^{W}$

Let $\{\underline{w}^1, \ldots, \underline{w}^{d_N}\} \subset N_S$ be a linearly independent basis of N_S. Here, d_N is the dimension of the null-space N_S. Then, for every $\underline{u} \in \tilde{D}(\Delta)$ one has

$$\underline{\underline{I}}^{C}_{S}\underline{u} = \sum_{\alpha=1}^{d_N} C_{\alpha}\underline{w}^{\alpha} \tag{D1}$$

where the set of coefficients $\{C_1, \ldots, C_{d_N}\}$ is the solution of the system of equations

$$\sum_{\alpha=1}^{d_N} C_\alpha \underline{w}^\alpha \bullet \underline{w}^\beta = \underline{u} \bullet \underline{w}^\beta, \beta = 1, \ldots, d_N \tag{D2}$$

If the basis $\{\underline{w}^1, \ldots, \underline{w}^{d_N}\}$ is orthonormal, then the matrix-system, in Eq. (D2), is the identity matrix.

On the other hand, for every $\underline{u} \in \tilde{D}(\Delta)$ one has

$$\underline{\underline{j}}^w \underline{u} = \sum_{\alpha=1}^{d_N} b_\alpha \underline{\underline{j}} \underline{w}^\alpha \tag{D3}$$

where the set of coefficients $\{b_1, \ldots, b_{d_N}\}$ is the solution of the system of equations

$$\sum_{\alpha=1}^{d_N} b_\alpha \underline{\underline{j}} \underline{w}^\alpha \bullet \underline{\underline{j}} \underline{w}^\beta = \underline{u} \bullet \underline{\underline{j}} \underline{w}^\beta, \beta = 1, \ldots, d_N \tag{D4}$$

As for $(\underline{\underline{j}}^w)^{-1} : \tilde{D}_{11}^w \to N_S$, let $\underline{t} \in \tilde{D}_{11}^w(\Delta)$ and $\underline{v} \equiv (\underline{\underline{j}}^w)^{-1} \underline{t} \in N_S$. Then

$$\underline{v} = \sum_{\alpha=1}^{d_N} c_\alpha \underline{w}^\alpha \tag{D5}$$

Furthermore, using $\underline{\underline{j}} \underline{v} = \underline{\underline{j}}^w \underline{v} = \underline{t}$,

$$\underline{\underline{j}} \underline{w}^\alpha \bullet \underline{v} = \underline{w}^\alpha \bullet \underline{\underline{j}} \underline{v} = \underline{w}^\alpha \bullet \underline{t}, \forall \alpha = 1, \ldots, d_N \tag{D6}$$

Therefore,

$$\sum_{\alpha=1}^{d_N} c_\alpha \underline{\underline{j}} \underline{w}^\alpha \bullet \underline{\underline{j}} \underline{w}^\beta = \underline{t} \bullet \underline{w}^\beta, \quad \beta = 1, \ldots, d_N \tag{D7}$$

Observe that the square-matrix, $d_N \times d_N$:

$$(\underline{\underline{j}} \underline{w}^\alpha \bullet \underline{\underline{j}} \underline{w}^\beta) \tag{D8}$$

is symmetric and positive definite in the space R^{d_N}. Therefore, application of Eqs. (D4) and (D7) yields the families of coefficients $\{b_1, \ldots, b_{d_N}\}$ and $\{c_1, \ldots, c_{d_N}\}$, respectively. Thereby, we mention that a criterion for the corner selection when applying dual-primal methods can be derived from the structure of the matrix of Eq. (D8). As a matter of the minimal cardinality of the set of such corners is d_N. This problem has been discussed previously by Lesoinne [10].

On the other hand, given any $\underline{u} \in \tilde{D}(\Delta)$ one has

$$\underline{\underline{k}}^w \underline{u} \in \tilde{D}_{11}^w(\Delta) = \underline{\underline{j}} N_S \subset \tilde{D}_{11}(\Delta) \tag{D9}$$

Therefore,

$$\underline{\underline{k}}^{w}\underline{u} = \sum_{\alpha=1}^{d_N} d_\alpha \underline{\underline{j}}\underline{w}^{\alpha} \tag{D10}$$

Using Eqs. (15.8) and (D9), it is seen that

$$\underline{\underline{j}}\underline{w}^{\beta} \bullet \underline{\underline{k}}^{w}\underline{u} = \underline{w}^{\beta} \bullet \underline{\underline{k}}^{w}\underline{u} = \underline{w}^{\beta} \bullet \underline{u}, \quad \beta = 1, \ldots, d_N \tag{D11}$$

Hence,

$$\sum_{\alpha=1}^{d_N} d_\alpha \underline{\underline{j}}\underline{w}^{\alpha} \bullet \underline{\underline{j}}\underline{w}^{\beta} = \underline{u} \bullet \underline{w}^{\beta} = C_\beta, \quad \beta = 1, \ldots, d_N \tag{D12}$$

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