A Beginner's Guide to *Mathematica*

Preliminary Edition

If $F'(x) = f(x)$,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

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Contents

Preface

About *A Beginner's Guide*

A Beginner's Guide is written for those students in education, mathematics, science, and engineering degree programs that require them to take calculus, differential equations, and linear algebra and is the ideal resource for those students who are required or desire to use Mathematica in the introductory calculus, differential equations, and linear algebra courses. Consequently, *A Beginner's Guide* is especially appropriate and written for students enrolled in these courses.

1. The text takes advantage of only those functions frequently used by beginning users and least likely to be adjusted in subsequent editions of the software. Occasionally, some code is presented to perform more sophisticated routines without much expanation. You should adjust these functions for your own purposes.

2. Each chapter concludes with several exercises. The exercises are designed to help you become more familiar with the mathematics as well as the computer algebra system.

Many subject-specific Mathematica texts are available for the more specialized areas of studies that students encounter in upper-division undergraduate and graduate courses.

A Beginner's Guide is especially useful for students enrolled at institutions that use both *Mathematica* and *Maple*. Editions of *A Beginner's Guide* for both *Maple* and *Mathematica* are available. The examples in each edition are nearly identical, while the software specific edition discusses the particular computer algebra system.

Cost of *A Beginner's Guide*

If you are a student enrolled in a mathematics course at Georgia Southern University, Statesboro, Georgia, there is no cost to use *A Beginner's Guide*. Otherwise, if you are located within the United States, the cost of using *A Beginner's Guide* is $6.00. Please include your e-mail address with your $6.00 so we can thank you when we receive it.

If you are not located within the continental United States, the cost of using *A Beginner's Guide* is a colorful postcard mailed with a pretty stamp from your area of the world. Please include your e-mail address on your postcard so we can thank you when we receive it. Please send your $6.00, if located within the United States, or postcard, if not located within the United States, to

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Funds received are used for scholarships. Postcards are displayed around our offices. In advance, we thank you for participating in the honor system. Contact us directly to make arrangements to distribute *A Beginner's Guide* to groups.

Remember that *A Beginner's Guide* is a work in progress. Please mail comments, errors, and suggestions for improvement to the above address or by e-mail to `jimbras@gsvms2.cc.gasou.edu`. Although this is a *Preliminary Edition* of *A Beginner's Guide*, several individuals have already offered feedback on various drafts. Thank you all for the time you have spent examining *A Beginner's Guide*.

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Chapter 1

Getting Started with Mathematica

We begin by introducing the essentials of Mathematica. The examples presented are taken from algebra, trigonometry, and calculus topics that you are familiar with to assist you in becoming acquainted with the Mathematica computer algebra system.

We assume that Mathematica has been correctly installed on the computer you are using. If you need to install Mathematica on your computer, please refer to the documentation that came with the Mathematica software package.

1.1 Running Mathematica

Let's begin our Mathematica session by starting Mathematica. Start Mathematica on your computer system. If you are not already in Mathematica or cannot start it, look for a folder or directory titled "Mathematica". Once you have located the Mathematica folder (or directory) open it. The directory typically looks like that shown in Figure 1.1. Using Windows or Macintosh mouse or keyboard commands, activate the Mathematica program by clicking or double-clicking appropriately.

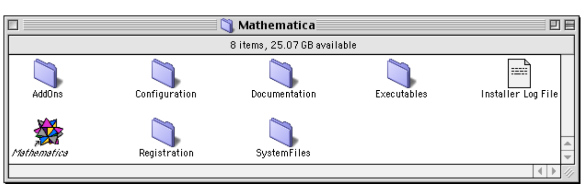


Figure 1.1: The Mathematica folder on a typical computer.

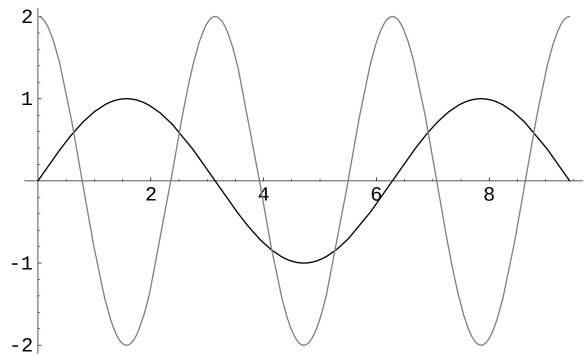


Figure 1.2: A two-dimensional plot

Remark. Throughout *A Beginner's Guide*, we assume that you are working in Mathematica. If you are not working in Mathematica, locate the Mathematica folder or directory as described previously and start the Mathematica program. Ask for assistance from your instructor or lab assistants, if necessary, to start the Mathematica program.

Once Mathematica has been started, computations can be carried out immediately. Mathematica commands are typed and the black horizontal line is replaced by the command, which is then evaluated by pressing **ENTER**. Note that pressing **ENTER** or **RETURN** evaluates commands and pressing **SHIFT-RETURN** yields a new line. Output is displayed below input. We illustrate some of the typical steps involved in working with Mathematica in the calculations that follow. In each case, we type the command and press **ENTER**. Mathematica evaluates the command, displays the result, and inserts a new horizontal line after the result. For example, entering

With some operating systems, **ENTER** evaluates commands and **RETURN** yields a new line

```
In[1]:= N[π, 50]
Out[1]= 3.1415926535897932384626433832795028841971693993751062.09749446
```

returns a 50-digit approximation of π.

The next calculation can then be typed and entered in the same manner as the first. For example, entering

In[2]:= **Solve[x^3 - 2x + 1 == 0]**

Out[2]= $\{\{x\to 1\},\{x\to \frac{1}{2}(-1-\sqrt{5})\},\{x\to \frac{1}{2}(-1+\sqrt{5})\}\}$

solves the equation $x^3 - 2x + 1 = 0$ for x. Subsequent calculations are entered in the same way. For example, entering

```
In[3]:= Plot[{Sin[x], 2Cos[2x]}, {x, 0, 3π},
          PlotStyle- > {GrayLevel[0], GrayLevel[0.5]}]
```

graphs the functions $y = \sin x$ and $y = 2\cos 2x$ and on the interval $[0, 3\pi]$ shown in Figure 1.2. Similarly, entering

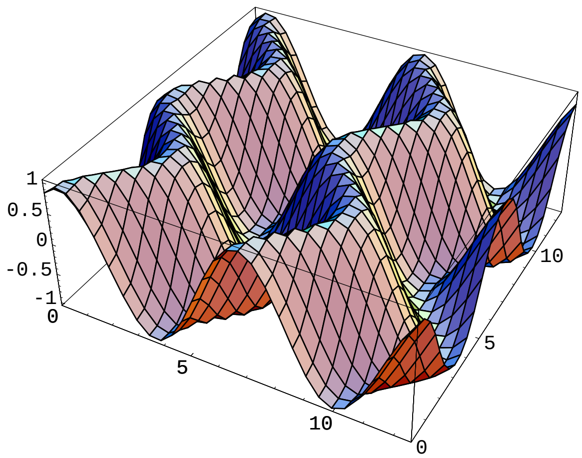


Figure 1.3: A three-dimensional plot

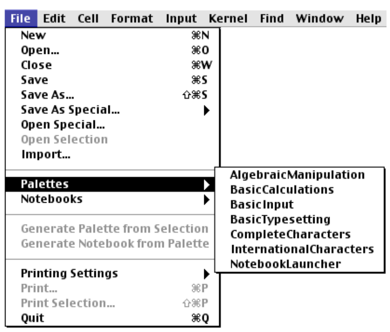


Figure 1.4: The file menu

```
In[4]:= Plot3D[Sin[x + Cos[y]], {x, 0, 4π}, {y, 0, 4π},
        PlotPoints- > {30, 30}]
```

graphs the function $z = \sin(x + \cos y))$ for $0 \leq x \leq 4\pi$ and $0 \leq y \leq 4\pi$ in Figure 1.3.

You can also enter input using the **Palettes** that are accessed from the **File** menu. (See Figure 1.4.) Two palettes are shown in Figure 1.5.

You can change how your input and output appear in your Mathematica notebook by going to **Cell** under the menu and selecting **Convert To**, **Display As**, **Default Input FormatType**, **Default Output FormatType**, or **Default Inline FormatType** as shown in Figure 1.6. Portions of **InputForm** or **StandardForm** can be selected, copied, and pasted elsewhere in your Mathematica notebook. Additional adjustments can be made from the Format menu. (See Figure 1.7.) This book includes real input and output from Mathematica. Appearances of input and output may vary depending on the version of Mathematica used, the fonts used to display input and output, the quality of the monitor,

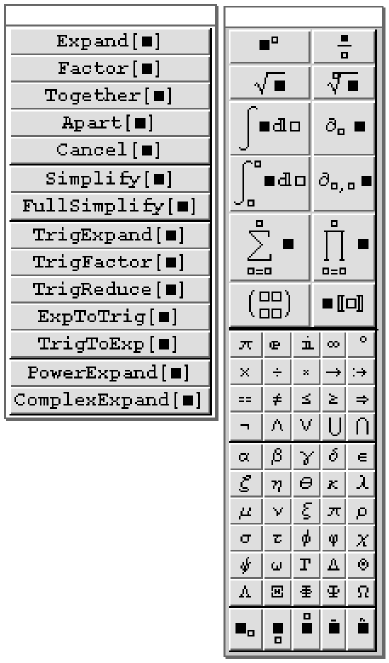


Figure 1.5: Two palettes that can be used to enter input

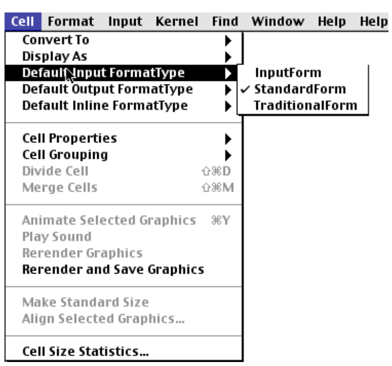


Figure 1.6: Available input and output display options

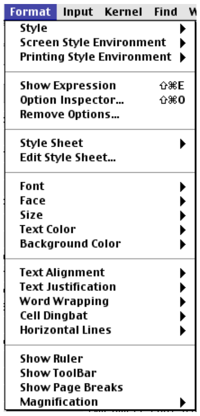


Figure 1.7: Additional formatting options

and the resolution and type of printer used to print the Mathematica worksheet: the results displayed on your computer may not be physically identical to those shown here.

Mathematica sessions are terminated by by selecting **Quit** from the **File** menu, or by using a keyboard shortcut as with other applications. They can be saved by referring to **Save** from the **File** menu.

Remark. Input and text regions in notebook interfaces can be edited. Editing input can create a notebook in which the mathematical output does not make sense in the sequence it appears. It is also possible to simply go into a notebook and alter input without doing any recalculation. This also creates misleading notebooks. Hence, common sense and caution should be used when editing the input regions of notebooks. Recalculating all commands in the notebook will clarify any confusion.

In order for the Mathematica user to take full advantage of the capabilities of this software, an understanding of its syntax is imperative. The goal of *A Beginner's Guide* is to introduce the reader to the Mathematica commands and sequences of commands most frequently used by beginning users in calculus, linear algebra, and differential equations courses. Although all of the rules of Mathematica syntax are far too numerous to list here, knowledge of the following five rules equips the beginner with the necessary tools to start using the Mathematica program with little trouble.

Five Basic Rules of Mathematica Syntax

1. The arguments of functions are given in brackets [...].
2. Every word of a built-in Mathematica function begins with a capital letter.
3. Multiplication is represented by a $*$ or space between characters . Enter `2*x*y` or `2x y`to evaluate $2xy$ *not* `2xy`.

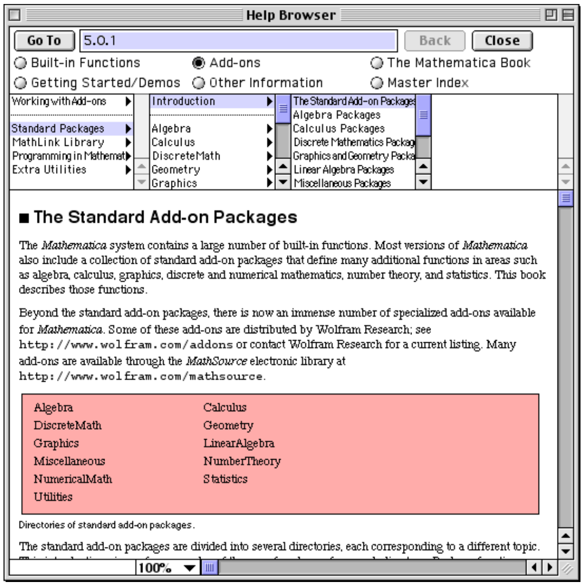


Figure 1.8: Standard Mathematica packages

4. Powers are denoted by a ˆ. Enter `(8*xˆ3)ˆ(1/3)` to evaluate $(8x^3)^{1/3} = 8^{1/3}(x^3)^{1/3} = 2x$ instead of `8xˆ1/3`, which returns `8x/3`.

5. Mathematica follows the order of operations *exactly*. Thus, entering `(1+x)ˆ1/x` returns $\frac{(1+x)^1}{x}$ while `(1+x)ˆ(1/x)` returns $(1+x)^{1/x}$.

   Remark. If you get no response or an incorrect response, you may have entered or executed the command incorrectly. In some cases, the amount of memory allocated to Mathematica can cause a crash; like people, Mathematica is not perfect and some errors can occur.

1.2 Loading Packages

Mathematica's modularity, which gives Mathematica a great deal of flexibility, helps minimize Mathematica's memory requirements. Nevertheless, although Mathematica contains many built-in functions that are loaded immediately when called, some other functions are contained in packages that must be loaded separately.

We access Mathematica's help facility from the menu to view a list of the available package groups shown in Figure 1.8.

A particularly useful group of packages is contained in the **Graphics** directory. (See Figure 1.9.)

We illustrate the use of the `FilledPlot` package, which is contained in the **Graphics**

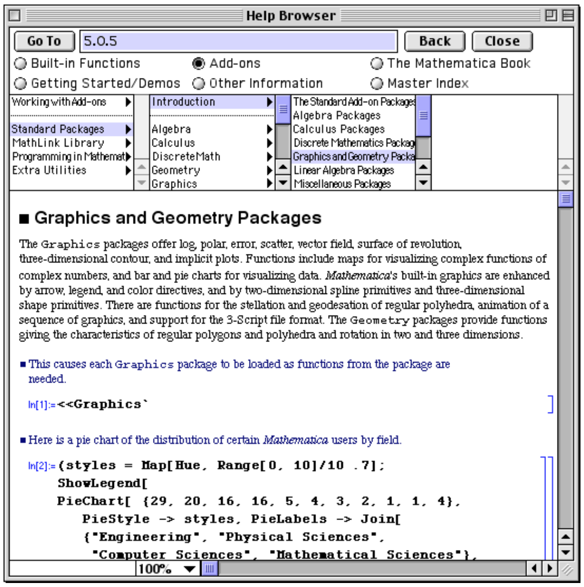


Figure 1.9: A description of the standard **Graphics** and **Geometry** packages

directory. (See Figure 1.10.)

We first load the functions contained in the **Graphics** directory and then use `FilledPlot` to shade the region between the graphs of $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$ shown in Figure 1.11.

```
In[5]:= << Graphics‘

In[6]:= FilledPlot[{Sin[x], Cos[x]}, {x, 0, 2π},
          AspectRatio- > Automatic]
```

1.3 Getting Help from Mathematica

Becoming competent with Mathematica can take a serious investment of time. Hopefully, messages that result from syntax errors are viewed lightheartedly. Ideally, instead of becoming frustrated, beginning Mathematica users will find it challenging and fun to locate the source of errors. Frequently, Mathematica's error messages indicate where the error(s) has (have) occurred. In this process, it is natural that one will become more proficient with Mathematica.

As we have seen previously, you can access Mathematica's help facility directly from the menu (see Figures 1.12 and 1.13).

From the main help window, you can search a variety of topics, as illustrated in Figures 1.14 and 1.15.

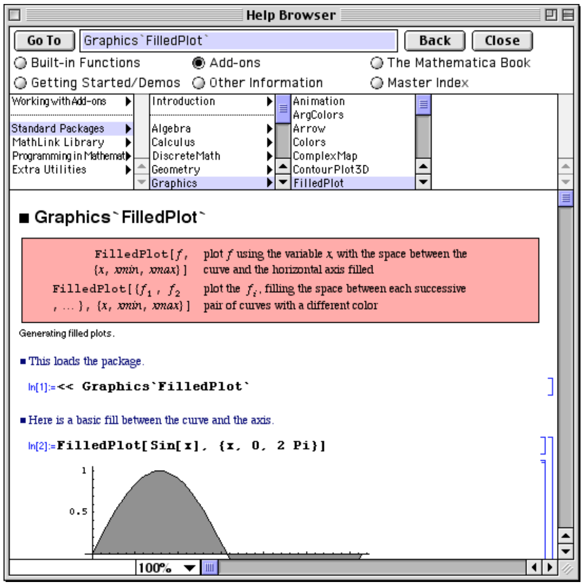


Figure 1.10: A description of the **FilledPlot** package

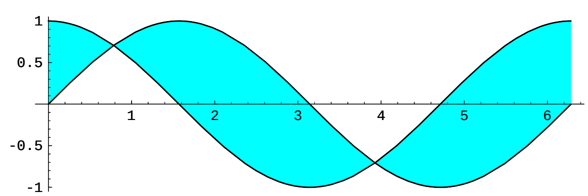


Figure 1.11: $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$

Figure 1.12: Accessing Mathematica help from the menu

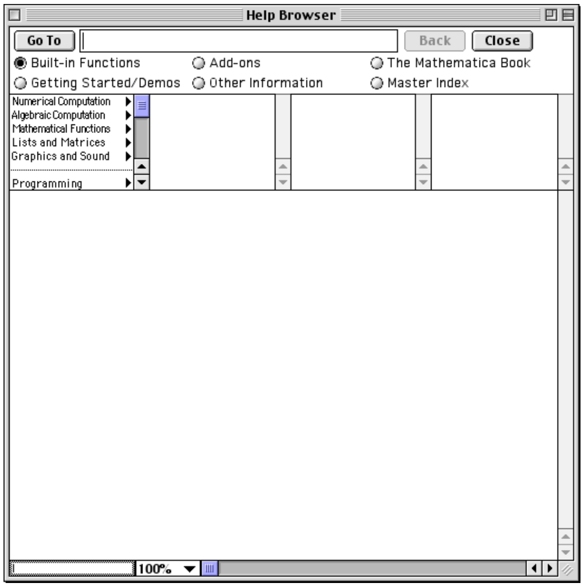


Figure 1.13: The main help window

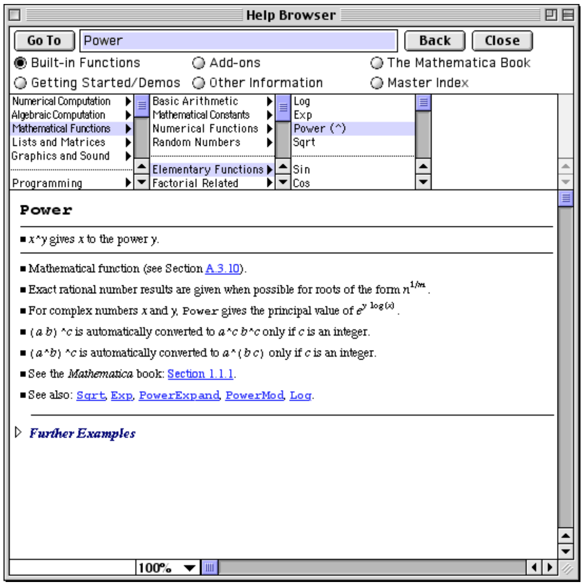


Figure 1.14: Help regarding `Power`

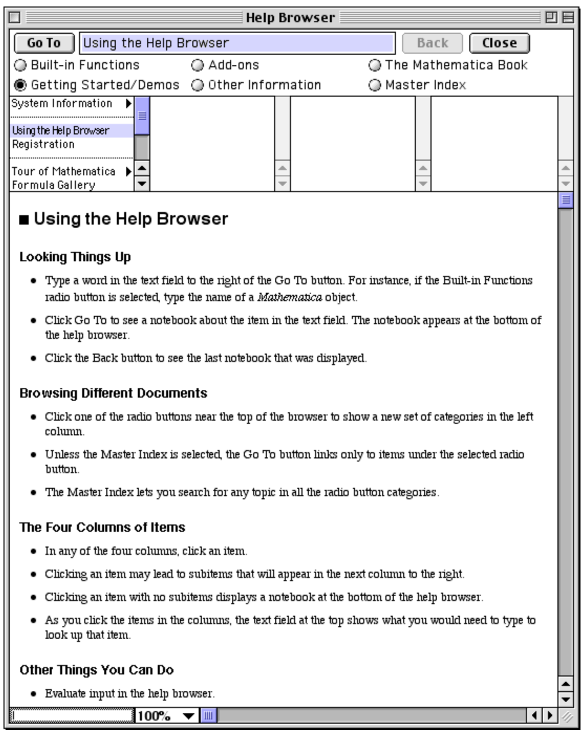


Figure 1.15: The Mathematica Help browser

You can obtain inline help using either `?` or `??`. Generally, `?Command` returns basic information regarding the syntax of `Command` while `??Command` returns the basic syntax as well as a list of associated options. For example, entering

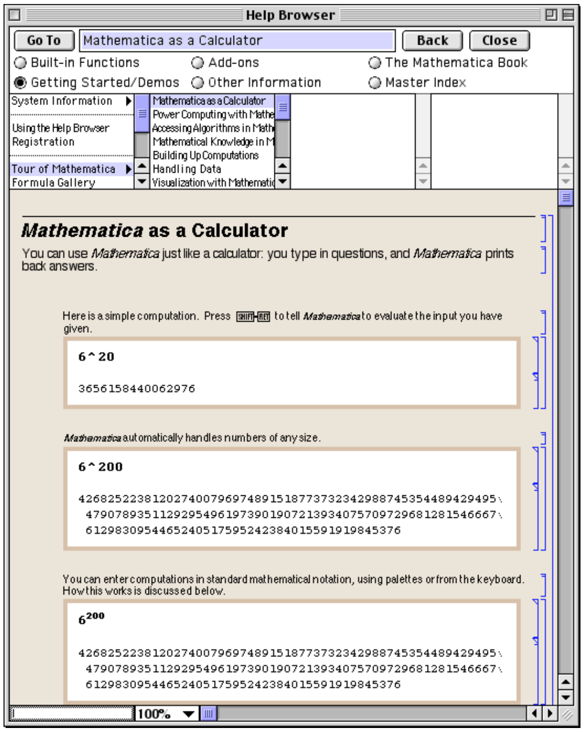
```
In[7]:= ?Plot
```

```
"Plot[f, x, xmin, xmax]generatesaplotof
   fasafunctionofxfromxmintoxmax.Plot[
   f1, f2, ..., x, xmin, xmax]plotsseveral
   functionsfi."
```

returns the basic syntax for the `Plot` function. Similarly, entering `?Plot3D` returns the basic syntax for the `Plot3D` function while entering `??Plot3D` returns the basic syntax and options for the `Plot3D` function.

```
?Plot3D

Plot3D[f, {x, xmin, xmax}, {y, ymin, ymax}] generates a three-
  dimensional plot of f as a function of x and y. Plot3D[
  {f, s}, {x, xmin, xmax}, {y, ymin, ymax}] generates
  a three-dimensional plot in which the height of the
  surface is specified by f, and the shading is specified by s.

?? Plot3D

Plot3D[f, {x, xmin, xmax}, {y, ymin, ymax}] generates a three-
  dimensional plot of f as a function of x and y. Plot3D[
  {f, s}, {x, xmin, xmax}, {y, ymin, ymax}] generates
  a three-dimensional plot in which the height of the
  surface is specified by f, and the shading is specified by s.

Attributes[Plot3D] = {HoldAll, Protected}

Options[Plot3D] =
 {AmbientLight -> GrayLevel[0], AspectRatio -> Automatic,
  Axes -> True, AxesEdge -> Automatic, AxesLabel -> None,
  AxesStyle -> Automatic, Background -> Automatic,
  Boxed -> True, BoxRatios -> {1, 1, 0.4},
  BoxStyle -> Automatic, ClipFill -> Automatic,
  ColorFunction -> Automatic, ColorOutput -> Automatic,
  Compiled -> True, DefaultColor -> Automatic,
  Epilog -> {}, FaceGrids -> None, HiddenSurface -> True,
  ImageSize -> Automatic, Lighting -> True,
  LightSources -> {{{1., 0., 1.}, RGBColor[1, 0, 0]},
    {{1., 1., 1.}, RGBColor[0, 1, 0]},
    {{0., 1., 1.}, RGBColor[0, 0, 1]}}, Mesh -> True,
  MeshStyle -> Automatic, Plot3Matrix -> Automatic,
  PlotLabel -> None, PlotPoints -> 15,
  PlotRange -> Automatic, PlotRegion -> Automatic,
  Prolog -> {}, Shading -> True, SphericalRegion -> False,
  Ticks -> Automatic, ViewCenter -> Automatic,
  ViewPoint -> {1.3, -2.399999999999999, 2.},
  ViewVertical -> {0., 0., 1.}, DefaultFont :> $DefaultFont,
  DisplayFunction :> $DisplayFunction,
  FormatType :> $FormatType, TextStyle :> $TextStyle}
```

The Help Browser offers additional tutorial and assistance for beginners in the **Getting Started/Demos** section. Many beginners find the topics included in the **Tour of Mathematica** particularly useful.



1.4 Numerical Calculations and Built-In Functions

1.4.1 Numerical Calculations

The basic arithmetic operations (addition, subtraction, multiplication, and division) are performed in the natural way with Mathematica. Mathematica *precisely* follows the standard order of operations.

1. "a plus b" is entered as $a + b$.
2. "a minus b" is entered as $a - b$.
3. "a times b" is entered as $a * b$.
4. "a divided by b" is entered as a/b. Generally, if a and b are integers, a/b results in the reduced fraction.
5. "a raised to the power b" is entered as $a \hat{}\, b$.

Example 1. Calculate (a) 121+542; (b) 3231−9876; (c) (−23)(76); (d) (22341)(832748)(387281); (e) $\frac{467}{31}$; and (f) $\frac{12315}{35}$.

Solution. These calculations are carried out in Figure 1.16. In (f), Mathematica simplifies the quotient because the numerator and denominator have a common factor of 5. In each case, the input is typed and then evaluated by pressing **ENTER**. □

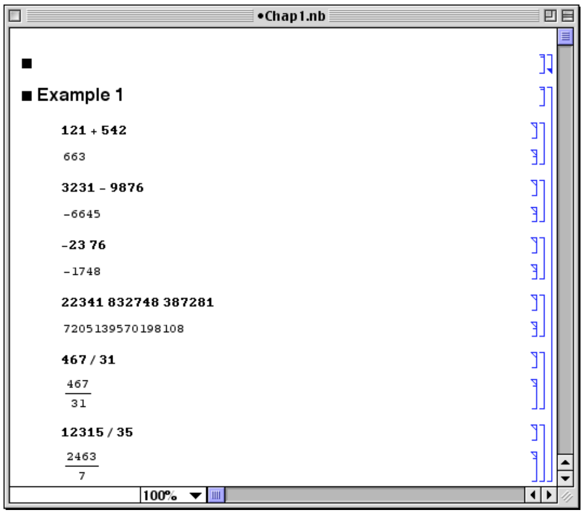


Figure 1.16: Arithmetic operations carried out in Mathematica.

The term $a^{n/m} = \sqrt[m]{a^n} = \left(\sqrt[m]{a}\right)^n$ is entered as `a^(n/m)`. For $n/m = 1/2$, the command `Sqrt[a]` can be used instead. Usually, the result is returned in unevaluated form but `N` can be used to obtain numerical approximations to virtually any degree of accuracy. With `N[expr,n]`, Mathematica yields a numerical approximation of `expr` to *n* digits of precision, if possible. At other times, `Simplify` can be used to produce the expected results.

Example 2. Compute (a) $\sqrt{27}$ and (b) $\sqrt[3]{8^2} = 8^{2/3}$.

Solution. (a) Mathematica automatically simplifies $\sqrt{27} = 3\sqrt{3}$.

```
In[8]:= Sqrt[27]
Out[8]= 3 √3
```

We use `N` to obtain an approximation of $\sqrt{27}$.

`N[number]` returns a numerical approximation of *number*.

```
In[9]:= N[Sqrt[27]]
Out[9]= 5.19615
```

(b) Mathematica automatically simplifies $8^{2/3}$.

```
In[10]:= 8^(2/3)
Out[10]= 4
```

□

When computing odd roots of negative numbers, Mathematica's results are surprising to the novice. Namely, Mathematica returns a complex number. We will see that this has important consequences when graphing certain functions.

Example 3. Calculate (a) $\frac{1}{3}\left(-\frac{27}{64}\right)^2$ and (b) $\left(-\frac{27}{64}\right)^{2/3}$.

Solution. (a) Because Mathematica follows the order of operations, `(-27/64)^2/3` first computes $(-27/64)^2$ and then divides the result by 3.

```
In[11]:= (-27/64)^2/3
Out[11]= 243/4096
```

(b) On the other hand, `(-27/64)^(2/3)` raises $-27/64$ to the $2/3$ power. Mathematica does not automatically simplify $\left(-\frac{27}{64}\right)^{2/3}$.

```
In[12]:= (-27/64)^(2/3)
Out[12]= 9/16 (-1)^(2/3)
```

However, when we use `N`, Mathematica returns the numerical version of the principal root of $\left(-\frac{27}{64}\right)^{2/3}$.

```
In[13]:= N[(-27/64)^(2/3)]
Out[13]= -0.28125 + 0.487139 i
```

To obtain the result

$$\left(-\frac{27}{64}\right)^{2/3} = \left(\sqrt[3]{\frac{-27}{64}}\right)^2 = \left(-\frac{3}{4}\right)^2 = \frac{9}{16},$$

which would be expected by most algebra and calculus students, we load the `RealOnly` package that is contained in the **Miscellaneous** directory. Then,

```
In[14]:= << Miscellaneous`RealOnly`

In[15]:= (-27/64)^(2/3)
Out[15]= 9/16
```

returns the result 9/16. □

1.4.2 Built-in Constants

Mathematica has built-in definitions of many commonly used constants. In particular, $e \approx 2.71828$ is denoted by `E`, $\pi \approx 3.14159$ is denoted by `Pi`, and $i = \sqrt{-1}$ is denoted by `I`. Usually, Mathematica performs complex arithmetic automatically.

Example 4. Entering

```
In[16]:= N[e, 50]
```

```
Out[16]= 2.7182818284590452353602874713526624977572470936999959
          7.4966967600000000000000000000000000000000000000000 10^8
```

returns a 50 digit approximation of e. Entering

```
In[17]:= N[π, 25]
Out[17]= 3.141592653589793238462643
```

returns a 25 digit approximation of π. Entering

```
In[18]:= (3 + i)/(4 - i)
Out[18]= 11/17 + 7 i/17
```

performs the division $(3 + i)/(4 - i)$ and writes the result in standard form.

1.4.3 Built-In Functions

Mathematica contains numerous mathematical functions.

Functions frequently encountered by beginning users include the exponential function, `Exp[x]`; the natural logarithm, `Log[x]`; the absolute value function, `Abs[x]`; the trigonometric functions `Sin[x]`, `Cos[x]`, `Tan[x]`, `Sec[x]`, `Csc[x]`, and `Cot[x]`; the inverse trigonometric functions `ArcSin[x]`, `ArcCos[x]`, `ArcTan[x]`, `ArcSec[x]`, `ArcCsc[x]`, and `ArcCot[x]`; the hyperbolic trigonometric functions `Sinh[x]`, `Cosh[x]`, and `Tanh[x]`; and their inverses `ArcSinh[x]`, `ArcCosh[x]`, and `ArcTanh[x]`. Generally, Mathematica tries to return an exact value unless otherwise specified with `N`.

Several examples of the natural logarithm and the exponential functions are given next. Mathematica often recognizes the properties associated with these functions and simplifies expressions accordingly.

Example 5. Entering

```
In[19]:= N[Exp[-5]]
Out[19]= 0.00673795
```

returns an approximation of $e^{-5} = 1/e^5$. Entering

```
In[20]:= Log[Exp[3]]
Out[20]= 3
```

computes $\ln e^3 = 3$. Entering

```
In[21]:= Exp[Log[4]]
Out[21]= 4
```

computes $e^{\ln 4} = 4$. Entering

```
In[22]:= Abs[-π]
Out[22]= π
```

`N[number]` returns a numerical approximation of *number*.
`Exp[x]` computes e^x. Enter `E` to compute $e \approx 2.718$.

`Log[x]` computes $\ln x$. $\ln x$ and e^x are inverse functions ($\ln e^x = x$ and $e^{\ln x} = x$) and Mathematica uses these properties when simplifying expressions involving these functions.

computes $|-\pi| = \pi$. Entering

`Abs[x]` returns the absolute value o

```
In[23]:= Abs[(3 + 2i)/(2 - 9i)]
```

Out[23]= $\sqrt{\frac{13}{85}}$

computes $|(3+2i)/(2-9i)|$. Entering

```
In[24]:= Sin[π/12]
```

Out[24]= $\frac{-1+\sqrt{3}}{2\sqrt{2}}$

computes the exact value of $\sin(\pi/12)$. Although Mathematica cannot compute the exact value of $\tan 1000$, entering

`N[number]` returns a numerical approximation of *number*.

```
In[25]:= N[Tan[1000]]
Out[25]= 1.47032
```

returns an approximation of $\tan 1000$. Similarly, entering

```
In[26]:= N[ArcSin[1/3]]
Out[26]= 0.339837
```

returns an approximation of $\sin^{-1}(1/3)$ and entering

```
In[27]:= ArcCos[2/3]//N
Out[27]= 0.841069
```

returns an approximation of $\cos^{-1}(2/3)$.

Mathematica is able to apply many identities that relate the trigonometric and exponential functions using the functions `TrigExpand`, `TrigFactor`, `TrigReduce`, `TrigToExp`, and `ExpToTrig`.

```
In[28]:= ?TrigExpand
"TrigExpand[expr]expandsouttrigonometric
   functionsinexpr."
In[29]:= ?TrigFactor
"TrigFactor[expr]factorstrigonometricfunctions
   inexpr."
In[30]:= ?TrigReduce
"TrigReduce[expr]rewritesproductsandpowers
   oftrigonometricfunctionsinexprinterms
   oftrigonometricfunctionswithcombinedarguments."
In[31]:= ?TrigToExp
"TrigToExp[expr]convertstrigonometricfunctions
   inexprtoexponentials."
In[32]:= ?ExpToTrig
"ExpToTrig[expr]convertsexponentialsinexpr
   totrigonometricfunctions."
```

Example 6. Mathematica does not automatically apply the identity $\sin^2 x + \cos^2 x = 1$.

```
In[33]:= Cos[x]^2 + Sin[x]^2
Out[33]= Cos[x]^2 + Sin[x]^2
```

To apply the identity, we use `Simplify`. Generally, `Simplify[expression]` attempts to simplify *expression*.

```
In[34]:= Simplify[Cos[x]^2 + Sin[x]^2]
Out[34]= 1
```

Use `TrigExpand` to multiply expressions or to rewrite trigonometric functions. In this case, entering

```
In[35]:= TrigExpand[Cos[3x]]
Out[35]= Cos[x]^3 - 3 Cos[x] Sin[x]^2
```

writes $\cos 3x$ in terms of trigonometric functions with argument x. We use the `TrigReduce` function to convert products to sums.

```
In[36]:= TrigReduce[Sin[3x] Cos[4x]]
Out[36]= 1/2 (-Sin[x] + Sin[7 x])
```

We use `TrigExpand` to write

```
In[37]:= TrigExpand[Cos[2x]]
Out[37]= Cos[x]^2 - Sin[x]^2
```

in terms of trigonometric functions with argument x. We use `ExpToTrig` convert exponential expressions to trigonometric expressions.

```
In[38]:= ExpToTrig[1/2(Exp[x] + Exp[-x])]
Out[38]= Cosh[x]
```

Similarly, we use `TrigToExp` to convert trigonometric expressions to exponential expressions.

```
In[39]:= TrigToExp[Sin[x]]
Out[39]= 1/2 i (e^(-i x) - e^(i x))
```

Usually, you can use `Simplify` to apply elementary identities.

```
In[40]:= Simplify[Tan[x]^2 + 1]
Out[40]= Sec[x]^2
```

1.5 The Basics

Beginning users of Mathematica typically need to acquire the ability to define, manipulate, and graph functions quickly. We illustrate how to perform these operations in the context of several examples from algebra, trigonometry, and calculus. Selected topics from calculus are discussed in more detail in Chapter 2. More sophisticated graphing techniques than those discussed here are introduced as they are needed.

Be careful to enter expressions precisely because Mathematica follows the order of operations in the standard order. Entering `xy` defines the symbol "xy". On the other hand, entering `x y` or `x*y` denotes the product of x and y, xy. Similarly, entering `(1+x)^1/x` computes

$$\frac{(1+x)^1}{x}$$

while entering `(1+x)^(1/x)` computes

$$(1+x)^{1/x}$$

because Mathematica follows the order of operations *exactly*.

For beginners, translations from standard mathematical notation to Mathematica can be particularly problematic when trigonometric functions are involved. For example, the expression $\sin x^2$ means that given x, square x, and compute the sine of the result. It is entered in Mathematica using `Sin[x^2]`. On the other hand, the expression $\sin^2 x$ means that given x, compute the sine of x, and square the result. The expression is entered in Mathematica using `Sin[x]^2`.

1.5.1 Elementary Operations on Functions of a Single Variable

In Mathematica, an elementary function of a single variable, $y = f(x) = \textit{expression in } x$, is typically defined using the form

```
f[x_]=expression in x.
```

Once the function $y = f(x)$ has been defined, a basic graph is generated with `Plot`:

```
Plot[f[x],{x,a,b}]
```

graphs $y = f(x)$ for $a \le x \le b$.

Example 7. Entering

```
In[41]:= Expand[(2x + 1)(3x - 1)(x - 1)]
Out[41]= 1 - 2 x - 5 x^2 + 6 x^3
```

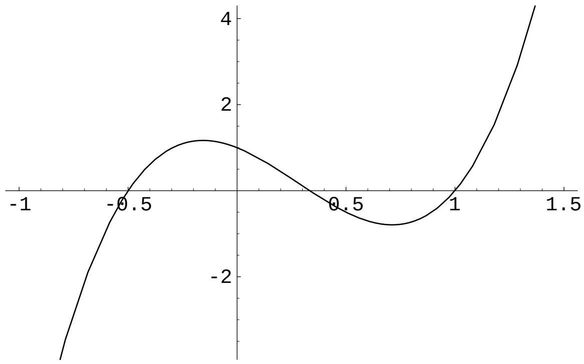


Figure 1.17: A basic graph of $f(x) = 6x^3 - 5x^2 - 2x + 1$ for $-1 \le x \le 3/2$

expands $(2x + 1)(3x - 1)(x - 1) = 6x^3 - 5x^2 - 2x + 1$. Entering

```
In[42]:= f[x_] = 6x^3 - 5x^2 - 2x + 1
Out[42]= 1 - 2 x - 5 x^2 + 6 x^3
```

defines $f(x) = 6x^3 - 5x^2 - 2x + 1$ and

```
In[43]:= Plot[f[x], {x, -1, 3/2}]
```

graphs $f(x) = 6x^3 - 5x^2 - 2x + 1$ for $-1 \le x \le 3/2$ in Figure 1.17.

For details regarding `Plot` and its options enter `?Plot` or `??Plot` or access help from the menu. (See Figure 1.18.) Frequently used `Plot` options are illustrated in the following examples.

Equations are solved with `Solve`:

```
Solve[lhs==rhs,x]
```

attempts to solve the equation $lhs = rhs$ for x;

```
Solve[{system of equations},{variables}]
```

attempts to solve *system of equations* for *variables*. In Mathematica, be sure to place a double equals sign (`==`) between the left and right-hand side of each equation.

Remember to include a semi-colon if you wish to suppress the result.

Example 8. Graph $g(x) = \frac{\sqrt{4-x^2}}{x^2-1}$.

Solution. In Mathematica, the square root function, $\sqrt{x}$, is represented by `Sqrt[x]`. We define $g(x)$ and then graph $g(x)$ for $-10 \le x \le 10$ with `Plot` in Figure 1.19. Observe that we obtain numerous error messages, although the resulting plot appears

Similarly, the absolute value function, $|x|$, is represented by `Abs[x]`.

We have chosen $[-10, 10]$ because it is a typical first choice for many students.

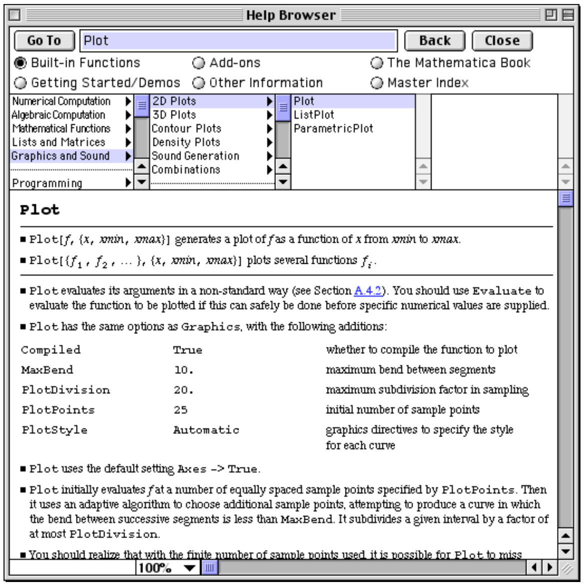


Figure 1.18: The `Plot` help window

reasonable. Notice that the plot is only shown for $-2 \le x \le 2$, not $-10 \le x \le 10$ as requested

In[44]:= **g[x_] = Sqrt[4 - x^2]/(x^2 - 1)**

Out[44]= $\dfrac{\sqrt{4 - x^2}}{-1 + x^2}$

In[45]:= **Plot[g[x], {x, -10, 10}]**

```
Plot :: "plnr" : "g[x] is not a machine-size real
     numberatx = -9.99999916666666699‘."

Plot :: "plnr" : "g[x] is not a machine-size real
     numberatx = -9.18866016854168421‘."

Plot :: "plnr" : "g[x] is not a machine-size real
     numberatx = -8.30382400281252586‘."

General :: "stop" : "FurtheroutputofPlot :: plnr

will be suppressed during this calculation."
```

Proceeding more carefully, we find the domain of $g(x)$. The domain of $g(x)$ consists of the values of x where the radicand is nonnegative *and* the denominator is not equal to zero. We solve $x^2 - 1 = 0$ with `Solve`. To solve $4 - x^2 \ge 0$ we use `InequalitySolve`, which is contained in the `InequalitySolve` package located in the **Algebra** directory.

In[46]:= **<< Algebra‘InequalitySolve‘**

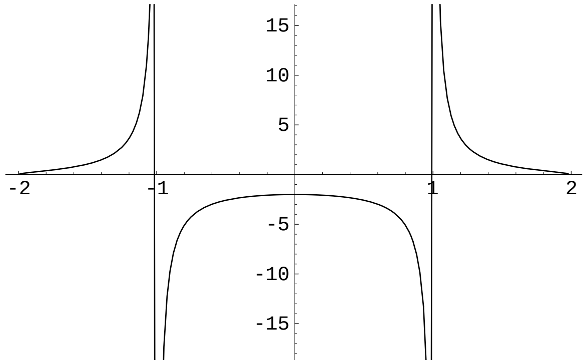


Figure 1.19: Mathematica attempts to choose an appropriate viewing window

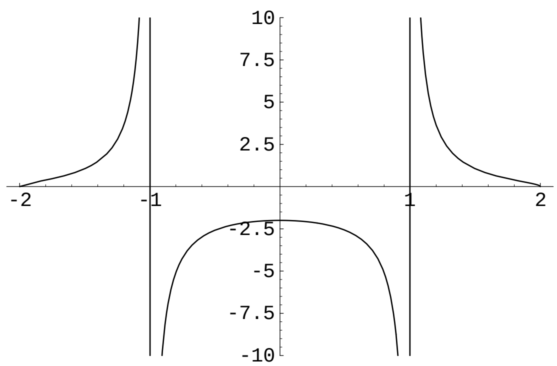


Figure 1.20: A good graph of $g(x)$

```
In[47]:= Solve[x^2 - 1 == 0]
Out[47]= {{x → -1}, {x → 1}}

In[48]:= InequalitySolve[4 - x^2 >= 0, x]
Out[48]= -2 ≤ x ≤ 2
```

We conclude that the domain is $[-2, -1) \cup (-1, 1) \cup (1, 2]$.

We now use `Plot` to graph $g(x)$ for $-2 \le x \le 2$ in Figure 1.20. We use the `PlotRange` option to specify that the x-values displayed correspond to $-2 \le x \le 2$ and the y-values displayed correspond to $-10 \le y \le 10$.

```
In[49]:= Plot[g[x], {x, -2, 2}, PlotRange- > {-10, 10}]
```

Of course, vertical lines are never the graphs of functions. In this example, the vertical lines correspond to the vertical asymptotes $x = 1$ and $x = -1$ of $g(x)$. □

Example 9. Solve

$$x = y - 1$$
$$x^2 = 2y + 6$$

for x and y.

Solution. We illustrate several techniques. First, we use `Solve` to solve the system for x and y.

```
In[50]:= Solve[{x == y - 1, x^2 == 2y + 6}]
Out[50]= {{y → -1, x → -2}, {y → 5, x → 4}}
```

By hand, we solve each equation for y and obtain $y = x + 1$ and $y = \frac{1}{2}(x^2 - 6)$. Then,

$$\begin{aligned} \frac{1}{2}(x^2 - 6) &= x + 1 \\ \frac{1}{2}x^2 - x - 4 &= 0 \\ x^2 - 2x - 8 &= 0 \\ (x - 4)(x + 2) &= 0 \end{aligned}$$

so $x = 4$ or $x = -2$. We perform the same steps using Mathematica with `Simplify` and `Factor`.

Remark. `Simplify[expression]` attempts to simplify *expression*; `Factor[expression]` attempts to factor *expression*.

```
In[51]:= s1 = Simplify[1/2(x^2 - 6) - (x + 1)]
Out[51]= -4 - x + x^2/2
In[52]:= Factor[s1]
Out[52]= 1/2 (-4 + x) (2 + x)
```

We also illustrate using the `Solve` function to solve $\frac{1}{2}(x^2 - 6) = x + 1$.

```
In[53]:= xvals = Solve[1/2(x^2 - 6) == (x + 1)]
Out[53]= {{x → -2}, {x → 4}}
```

Finally, we use `Plot` to graph $y = x + 1$ and $y = \frac{1}{2}(x^2 - 6)$ together for $-3 \le x \le 5$ in Figure 1.21. We illustrate the use of the `PlotStyle` and `AspectRatio` options. `PlotStyle->{GrayLevel[0],GrayLevel[0.3]}` specifies that the first plot be generated in black and the second in gray. `AspectRatio->Automatic` specifies that the plot be generated to scale.

```
In[54]:= Plot[{x + 1, 1/2(x^2 - 6)}, {x, -3, 5},
           PlotStyle- > {GrayLevel[0], GrayLevel[0.3]},
           AspectRatio- > Automatic]
```

In the figure, we see that $x + 1 \ge \frac{1}{2}(x^2 - 6)$ for $-2 \le x \le 4$. Thus, the area of the region bounded by the graphs of $y = x + 1$ and $y = \frac{1}{2}(x^2 - 6)$ is

$$\int_{-2}^{4} \left[(x + 1) - \frac{1}{2}(x^2 - 6) \right] dx.$$

Generally, `Integrate[f[x],{x,a,b}]` attempts to evaluate $\int_a^b f(x)\,dx$. Thus,

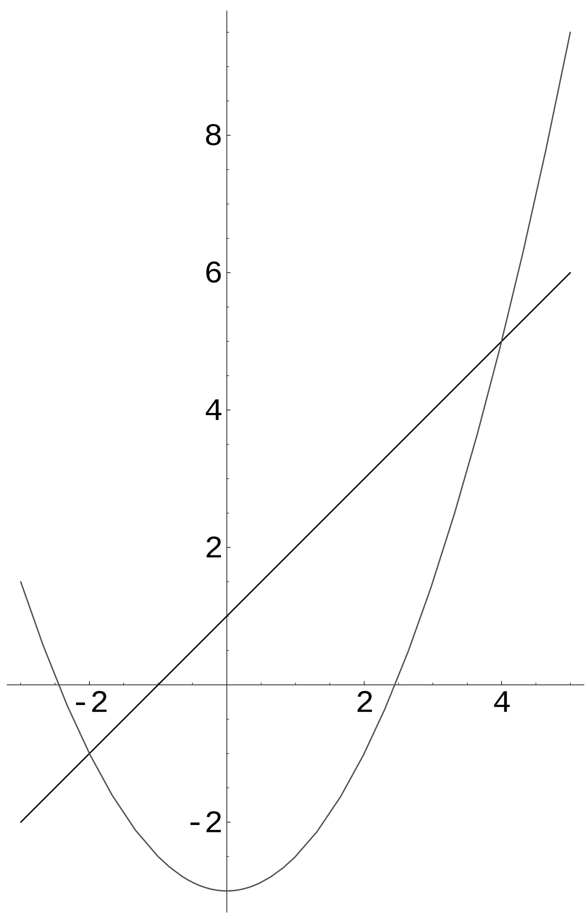


Figure 1.21: Graphs of $y = x + 1$ and $y = \frac{1}{2}(x^2 - 6)$

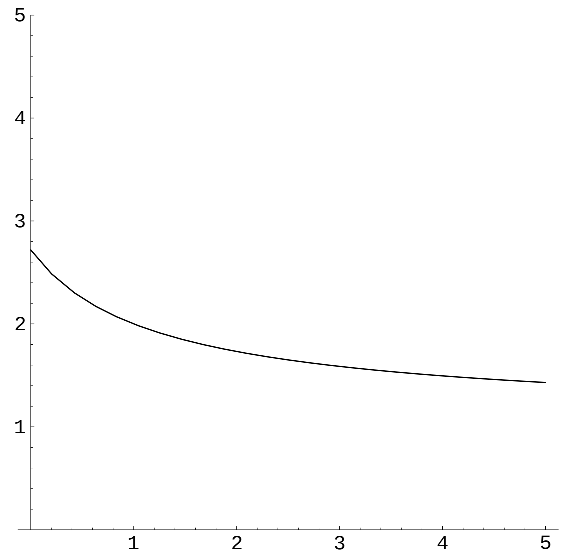


Figure 1.22: Graph of $f(x)$

```
In[55]:= Integrate[(x + 1) - 1/2(x^2 - 6), {x, -2, 4}]
Out[55]= 18
```

computes $\int_{-2}^{4} \left[(x+1) - \frac{1}{2}(x^2 - 6)\right] dx = 18.$ □

Example 10. Graph

$$f(x) = (1+x)^{1/x}$$

for $0 < x \leq 5$.

Solution. The domain of $f(x)$ is $(0, \infty)$. After defining $f(x)$, we use `Plot` to graph $f(x)$ for $0 < x \leq 5$ in Figure 1.22.

```
In[56]:= f[x_] = (1 + x)^(1/x)
Out[56]= (1 + x)^(1/x)
```

```
In[57]:= Plot[f[x], {x, 0, 5}, PlotRange- > {0, 5},
           AspectRatio- > Automatic]
```

`Table[a[k],{k,n,m}]` generates the sequence $a_n, a_{n+1}, a_{n+2} \dots a_{m-1}, a_m$.

Even though $f(x)$ is undefined if $x = 0$, Mathematica does not complain when we instruct it to begin the plot at $x = 0$. In the plot, we see that $x = 0$ does not appear to be an asymptote and appears as though $\lim_{x \to 0^+} f(x)$ exists. We use `Table` to compute the value of $f(x)$ for $x = 1, 1/10, \dots, 1/100000$– values of x ”near” $x = 0$.

```
In[58]:= Table[{10^(-k)//N, f[10^(-k)//N]}, {k, 0, 5}]
```

```
Out[58]= {{1., 2.}, {0.1, 2.59374},
          {0.01, 2.70481},
          {0.001, 2.71692},
          {0.0001, 2.71815},
          {0.00001, 2.71827}}
```

Notice that the function values appear to be getting closer to $e \approx 2.718$. In fact, in calculus, we learn that

$$\lim_{x \to 0^+} (1 + x)^{1/x} = e.$$

Mathematica is able to calculate this limit with `Limit`.

`Limit[f[x],x->a]` attempts to compute $\lim_{x \to a} f(x)$

```
In[59]:= Limit[f[x], x- > 0]
Out[59]= e
```

□

Example 11. Let $f(x) = 6x^3 - 5x^2 - 2x + 1$. (a) Evaluate $f(2)$ and $f(1)$. (b) Compute and simplify $\frac{f(1+h)-f(1)}{h}$. (c) Find $\lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$. (d) Solve $f(x) = 0$. (e) Graph $f(x)$. (f) Graph $f(x)$ together with the line tangent to the graph of $f(x)$ at the point with x-coordinate $x = 1$. (g) Find $f'(x)$. (h) Solve $f'(x) = 0$ exactly and numerically. (i) Find $\int f(x)\,dx$. (j) Evaluate $\int_{-1/2}^{1/3} f(x)\,dx$.

Solution. After defining $f(x)$, we compute $f(2)$ and $f(1)$.

```
In[60]:= f[x_] = 6x^3 - 5x^2 - 2x + 1
Out[60]= 1 - 2 x - 5 x^2 + 6 x^3

In[61]:= f[2]
Out[61]= 25

In[62]:= f[1]
Out[62]= 0
```

We compute $\frac{f(1+h)-f(1)}{h}$ naming the result `s1`.

```
In[63]:= s1 = (f[1 + h] - f[1])/h
Out[63]= (1 - 2 (1 + h) - 5 (1 + h)^2 + 6 (1 + h)^3) / h
```

`s1` is then simplified with `Simplify` and named `s2`.

```
In[64]:= s2 = Simplify[s1]
Out[64]= 6 + 13 h + 6 h^2
```

The limit of `s2` as $h \to 0$ is computed with `Limit`.

```
In[65]:= Limit[s2, h- > 0]
Out[65]= 6
```

Note that the entire computation can be combined into a single command.

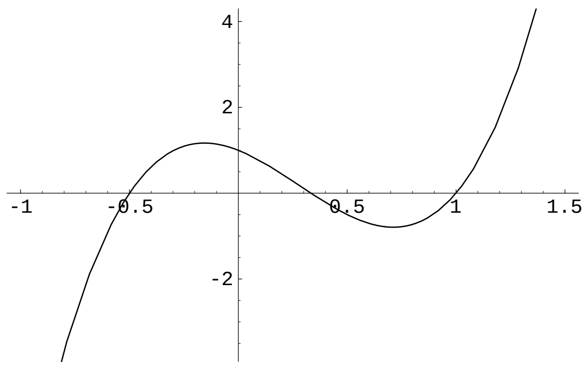


Figure 1.23: Graph of $f(x)$

```
In[66]:= Limit[(f[1+h] - f[1])/h, h-> 0]
Out[66]= 6
```

Using `Factor`, we see that the zeros of $f(x)$ are $x = 1/3$, 1, and $-1/2$.

```
In[67]:= Factor[f[x]]
Out[67]= (-1 + x) (1 + 2 x) (-1 + 3 x)
```

We confirm by solving $f(x) = 0$ with `Solve`.

```
In[68]:= Solve[f[x] == 0]
```

Out[68]= $\left\{\left\{x \to -\frac{1}{2}\right\}, \left\{x \to \frac{1}{3}\right\}, \{x \to 1\}\right\}$

A basic graph of $f(x)$ is generated with `Plot` in Figure 1.23.

```
In[69]:= Plot[f[x], {x, -1, 3/2}]
```

Because $f(1) = 0$ and the slope of the line tangent to the graph of $f(x)$ at the point with x-coordinate $x = 1$ is 6, an equation of the line tangent to the graph of $f(x)$ at the point $(1, 0)$ is $y - 0 = 6(x - 1)$ or $y = 6(x - 1)$. We graph $f(x)$ and $y = 6(x - 1)$ with `Plot` in Figure 1.24.

```
In[70]:= Plot[{f[x], 6(x - 1)}, {x, -3/2, 3/2},
           PlotStyle-> {GrayLevel[0], GrayLevel[0.3]}]
```

`f'[x]` computes $dy/dx = f'(x)$.

We find $f'(x)$ with `'`.

```
In[71]:= df = f'[x]
```

Out[71]= $-2 - 10\ x + 18\ x^2$

We use `Solve` to find that $f'(x) = 0$ if $x = \frac{5}{18} \pm \frac{1}{18}\sqrt{61}$ and name these values `df0`.

```
In[72]:= df0 = Solve[f'[x] == 0]
```

Out[72]= $\left\{\left\{x \to \frac{1}{18}\left(5 - \sqrt{61}\right)\right\}, \left\{x \to \frac{1}{18}\left(5 + \sqrt{61}\right)\right\}\right\}$

`N[number]` returns an approximation of *number*.

We then use `N` to obtain approximations of the exact values.

```
In[73]:= N[df0]
```

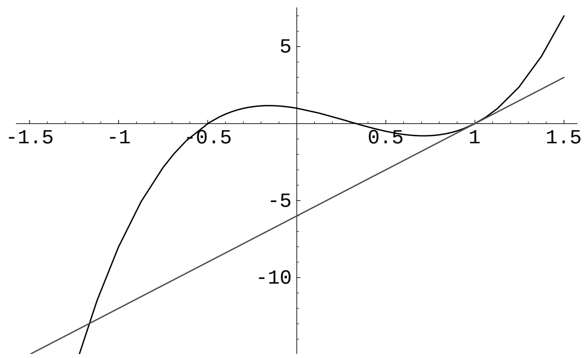


Figure 1.24: Graphs of $f(x)$ and $y = 6(x - 1)$

```
Out[73]= {{x → -0.156125},
          {x → 0.711681}}
```

Finally, we use `Integrate` to evaluate $\int f(x)\,dx$ and $\int_{-1/2}^{1/3} f(x)\,dx$.

`Integrate[f[x],x]` attempts to evaluate $\int f(x), dx$ while `Integrate[f[x],x,a,b]` attempts to compute $\int_a^b f(x)\,dx$.

```
In[74]:= Integrate[f[x], x]
```

Out[74]= $x - x^2 - \frac{5\ x^3}{3} + \frac{3\ x^4}{2}$

```
In[75]:= i0 = Integrate[f[x], {x, -1/2, 1/3}]
```

Out[75]= $\frac{1625}{2592}$

```
In[76]:= N[i0]
Out[76]= 0.626929
```

The results indicate that $\int f(x)\,dx = \frac{3}{2}x^4 - \frac{5}{3}x^3 - x^2 + x + C$ and $\int_{-1/2}^{1/3} f(x)\,dx = \frac{1625}{2592} \approx 0.6269$. □

Example 12. Let

$$f(t) = \frac{t^3}{t^2 - 1}.$$

(a) Compute and simplify $\lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$. (b) Compute $f'(t)$ and $f''(t)$. (c) Graph $f(t)$, $f'(t)$, and $f''(t)$.

Solution. After defining $f(t)$, we compute $\frac{f(t+h)-f(t)}{h}$, naming the result `s1`.

```
In[77]:= f[t_] = t^3/(t^2 - 1)
```

Out[77]= $\frac{t^3}{-1 + t^2}$

```
In[78]:= s1 = (f[t + h] - f[t])/h
```

Out[78]= $\frac{-\frac{t^3}{-1+t^2} + \frac{(h+t)^3}{-1+(h+t)^2}}{h}$

`s1` is simplified with `Together`. (`Together[fraction]` writes complex fractions as a single fraction.)

```
In[79]:= s2 = Together[s1]
Out[79]= (-h^2 - 3 h t - 3 t^2 + h^2 t^2 + 2 h t^3 + t^4)/((-1 + t^2) (-1 + h^2 + 2 h t + t^2))
```

`Limit` is used to compute $\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$. The result is $f'(t)$.

```
In[80]:= Limit[s2, h- > 0]
Out[80]= (-3 t^2 + t^4)/(-1 + t^2)^2
```

We obtain the same result using `Together` and `'` in `df`.

```
In[81]:= df = f'[t]//Together
Out[81]= (-3 t^2 + t^4)/(-1 + t^2)^2
```

We solve $f'(t) = 0$ with `Solve`.

```
In[82]:= Solve[df == 0]
Out[82]= {{t → 0}, {t → 0}, {t → -√3}, {t → √3}}
```

We compute and simplify $f''(t)$ with `Together` and `''` and then use `Solve` to solve $f''(t) = 0$.

```
In[83]:= d2f = f''[t]//Together
Out[83]= 2 (3 t + t^3)/(-1 + t^2)^3
In[84]:= Solve[d2f == 0]
Out[84]= {{t → 0}, {t → -i √3}, {t → i √3}}
```

Finally, we use `Plot` to graph $f(t)$, $f'(t)$, and $f''(t)$ together in Figure 1.25, illustrating the use of the `PlotRange`, `PlotStyle`, and `AspectRatio` options.

```
In[85]:= Plot[{f[t], df, d2f},
           {t, -6, 6}, PlotRange- > {-5, 5}, PlotStyle- >
            {GrayLevel[0], Dashing[{0.01}], GrayLevel[0.3]},
           AspectRatio- > Automatic]
```

□

Be especially careful when manipulating trigonometric functions.

Example 13. Let $f(\theta) = \sin 2\theta + 2\cos\theta$, $0 \le \theta \le 2\pi$. (a) Solve $f'(\theta) = 0$. (b) Graph $f(\theta)$ and $f'(\theta)$.

`D[f[x],x]` computes $f'(x)$; `D[f[x],{x,n}]` computes $f^{(n)}(x)$.

Solution. After defining $f(\theta)$, we use `D` to compute $f'(\theta)$ and then use `Solve` to solve $f'(\theta) = 0$.

```
In[86]:= f[θ_] = Sin[2θ] + 2Cos[θ]
Out[86]= 2 Cos[θ] + Sin[2 θ]
```

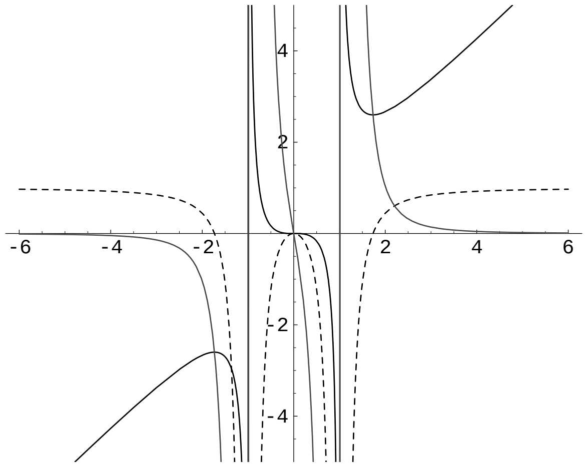


Figure 1.25: Graphs of $f(t)$, $f'(t)$, and $f''(t)$

```
In[87]:= df = D[f[θ], θ]
Out[87]= 2 Cos[2 θ] - 2 Sin[θ]
```

```
In[88]:= Solve[df == 0, θ]
Solve :: "ifun" : "Inversefunctionsarebeingused
     bySolve, so some solutions may not be found."
```

Out[88]= $\left\{\left\{\theta \to -\frac{\pi}{2}\right\}, \left\{\theta \to \frac{\pi}{6}\right\}, \left\{\theta \to \frac{5\ \pi}{6}\right\}\right\}$

Notice that $-\pi/2$ is not between 0 and 2π. Moreover, $\pi/6$ and $5\pi/6$ are *not* the only solutions of $f'(\theta) = 0$ between 0 and 2π. Proceeding by hand, we use the identity $\cos 2\theta = 1 - 2\sin^2\theta$ and factor:

$$\begin{aligned} 2\cos 2\theta - 2\sin\theta &= 0 \\ 1 - 2\sin^2\theta - \sin\theta &= 0 \\ 2\sin^2\theta + \sin\theta - 1 &= 0 \\ (2\sin\theta - 1)(\sin\theta + 1) &= 0 \end{aligned}$$

so $\sin\theta = 1/2$ or $\sin\theta = -1$. Because we are assuming that $0 \le \theta \le 2\pi$, we obtain the solutions $\theta = \pi/6$, $5\pi/6$, or $3\pi/2$. We perform the same steps with Mathematica.

expression /. x->y+ replaces all occurrences of *x* in *expression* by *y*.

```
In[89]:= s1 = TrigExpand[df]
Out[89]= 2 Cos[θ]^2 - 2 Sin[θ] - 2 Sin[θ]^2
```

```
In[90]:= s2 = s1/. Cos[θ]^2-> 1 - Sin[θ]^2
Out[90]= -2 Sin[θ] - 2 Sin[θ]^2 + 2 (1 - Sin[θ]^2)
```

```
In[91]:= Factor[s2]
Out[91]= -2 (1 + Sin[θ]) (-1 + 2 Sin[θ])
```

Finally, we graph $f(\theta)$ and $f'(\theta)$ with `Plot` in Figure 1.26. Note that the plot is drawn to scale because we include the option `AspectRatio->Automatic`.

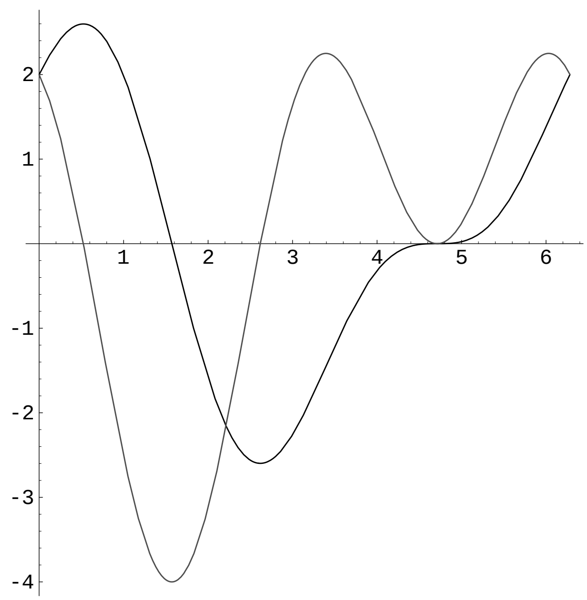


Figure 1.26: Graphs of $f(\theta)$ and $f'(\theta)$

```
In[92]:= Plot[{f[θ], df}, {θ, 0, 2π},
           PlotStyle-> {GrayLevel[0], GrayLevel[0.3]},
           AspectRatio-> Automatic]
```

□

If `Solve` is unsuccessful in solving an equation or numerical results are desired, `FindRoot[equation,{x,a}]` attempts to numerically solve *equation* for x near $x \approx a$.

Example 14. Find the first three nonnegative solutions of $x = \tan x$.

Solution. We attempt to solve $x = \tan x$ with `Solve`.

```
In[93]:= Solve[x == Tan[x], x]
Solve :: "tdep" : "Theequationsappeartoinvolve
     transcendentalfunctionsofthevariablesin
     anessentiallynon - algebraicway."
Out[93]= Solve[x == Tan[x], x]
```

We next graph $y = x$ and $y = \tan x$ together in Figure 1.27.

```
In[94]:= Plot[{x, Tan[x]}, {x, 0, 4π}, PlotRange-> {-4π, 4π},
           PlotStyle-> {GrayLevel[0], GrayLevel[0.3]}]
```

Remember that vertical lines are never the graphs of functions. In this case, the represent the vertical asymptotes at odd multiples of $\pi/2$.

In the graph, we see that $x = 0$ is a solution. This is confirmed with `FindRoot`.

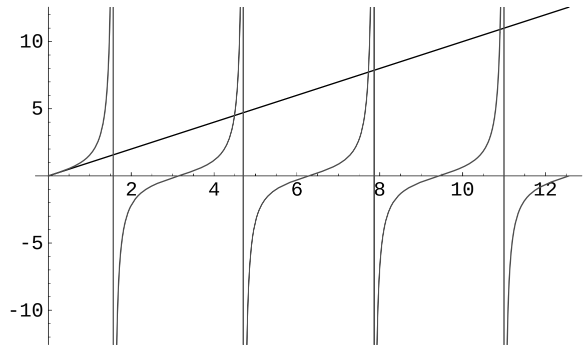


Figure 1.27: $y = x$ and $y = \tan x$

```
In[95]:= FindRoot[x == Tan[x], {x, 0}]
Out[95]= {x → 0.}
```

The second solution is near 4 while the third solution is near 7. Using `FindRoot` together with these initial approximations locates the second two solutions.

```
In[96]:= FindRoot[x == Tan[x], {x, 4}]
Out[96]= {x → 4.49341}
```

```
In[97]:= FindRoot[x == Tan[x], {x, 7}]
Out[97]= {x → 7.72525}
```

□

1.5.2 Elementary Parametric and Polar Plots

To graph the parametric equations $x = x(t)$, $y = y(t)$, $a \le t \le b$, use

```
ParametricPlot[{x[t],y[t]},{t,a,b}]
```

(see Figure 1.28) and to graph the polar function $r = r(\theta)$, $\alpha \le \theta \le \beta$, use

```
PolarPlot[r[theta],{theta,alpha,beta}].
```

(See Figure 1.29.) The `PolarPlot` function is contained in the `Graphics` package which is located in the **Graphics** directory.

Example 15. Graph the parametric equations

$$\begin{array}{l} x = t + \sin 2t \\ y = t + \sin 3t \end{array}, \quad -2\pi \le t \le 2\pi.$$

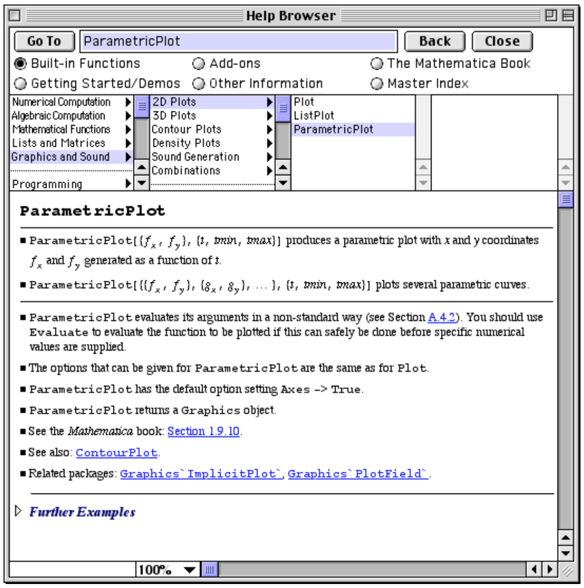


Figure 1.28: Mathematica's help for parametric plots

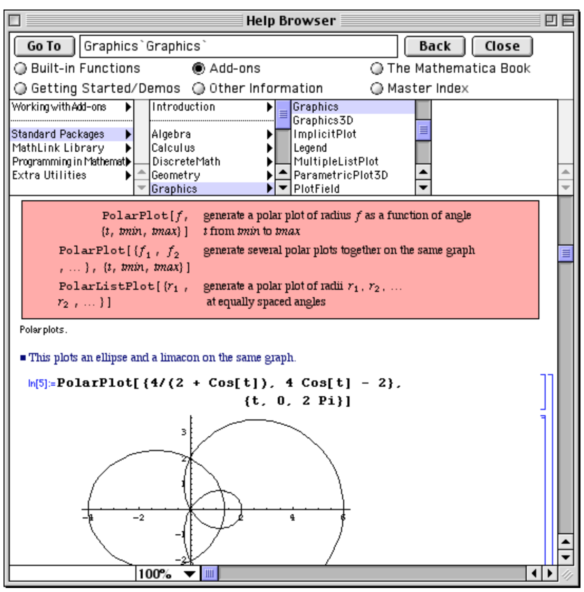


Figure 1.29: Mathematica's help for polar plots

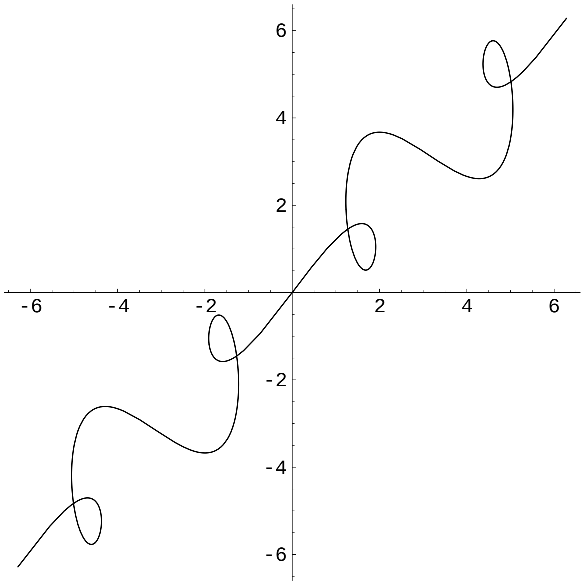


Figure 1.30: $(x(t), y(t))$, $-2\pi \le t \le 2\pi$

Solution. After defining x and y, we use `ParametricPlot` to graph the parametric equations in Figure 1.30.

```
In[98]:= x[t_] = t + Sin[2t];
         y[t_] = t + Sin[3t];
         ParametricPlot[
           {x[t], y[t]}, {t, -2π, 2π}, AspectRatio -> Automatic]
```

In calculus we learn that

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \text{and} \qquad y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'/dt}{dx/dt}.$$

For illustrative purposes, we perform these computations using `'` and `Simplify`. We compute dx/dt and dy/dt in `dx` and `dy`, respectively. $y' = dy/dx$ is formed in `dydx`.

```
In[99]:= dx = x'[t]

         dy = y'[t]

         dydx = dy/dx
Out[99]= 1 + 2 Cos[2 t]
Out[99]= 1 + 3 Cos[3 t]
         1 + 3 Cos[3 t]
Out[99]= --------------
         1 + 2 Cos[2 t]
```

Next, we compute and simplify dy'/dt in `d2ydx`.

```
In[100]:= d2ydx = Simplify[D[dydx, t]]
```

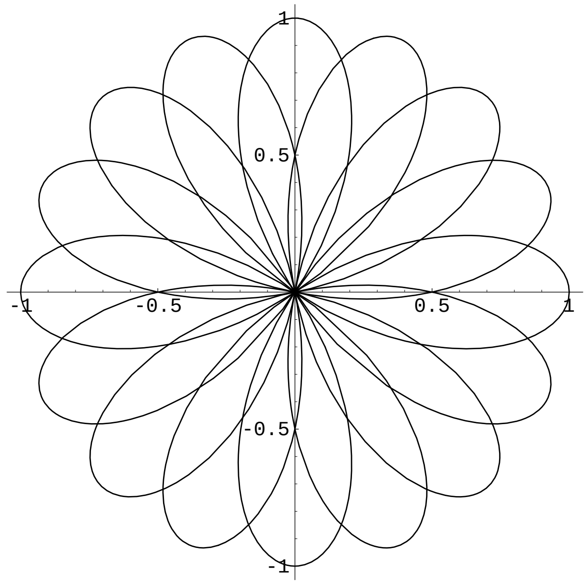


Figure 1.31: Graph of $r = \cos(8\pi/3)$

```
Out[100]= (-15 Sin[t] + 4 Sin[2 t] - 3 (3 Sin[3 t] + Sin[5 t])) / (1 + 2 Cos[2 t])^2
```

Finally, $y'' = d^2y/dx^2$ is computed and simplified in `d2ydx2`.

```
In[101]:= d2ydx2 = Simplify[d2ydx/dx]
Out[101]= (-15 Sin[t] + 4 Sin[2 t] - 3 (3 Sin[3 t] + Sin[5 t])) / (1 + 2 Cos[2 t])^3
```

□

Example 16. Graph $r = \cos(8\theta/3)$ for $0 \leq \theta \leq 6\pi$.

Solution. After loading the `Graphics` package and defining r, we use `PolarPlot` to graph the polar equation in Figure 1.31.

```
In[102]:= << Graphics`Graphics`

          r[θ_] = Cos[8θ/3];
          PolarPlot[r[θ], {θ, 0, 6π}, AspectRatio -> Automatic]
```

In calculus, we learn that the length of the graph of the polar equation $r = f(\theta)$, $\alpha \leq \theta \leq \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

We illustrate how this computation can be carried out with Mathematica.

After computing $dr/d\theta$ in `dr`, we compute and simplify $r^2 + (dr/d\theta)^2$ in `s1`.

```
In[103]:= dr = r'[θ]
Out[103]= -8/3 Sin[8 θ/3]

In[104]:= s1 = Simplify[r[θ]^2 + dr^2]
Out[104]= 1/18 (73 - 55 Cos[16 θ/3])
```

We then compute $L = \int_0^{6\pi} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$ in `s2`. However, the result is given in terms of the `EllipticE` function, a function not typically encountered by beginning users.

```
In[105]:= s2 = Integrate[Sqrt[s1], {θ, 0, 6π}]
Out[105]= 12 EllipticE[-55/9]
```

`N` is used to obtain a more meaningful approximation.

```
In[106]:= N[s2]
Out[106]= 36.3669
```

□

1.5.3 Three-Dimensional and Contour Plots; Graphing Equations

An elementary function of two variables, $z = f(x, y) = expression\ in\ x\ and\ y$, is typically defined using the form

```
f[x_,y_]=expression in x and y.
```

Once a function has been defined, a basic graph is generated with `Plot3D`:

```
Plot3D[f[x,y],{x,a,b},{y,c,d}]
```

graphs $f(x, y)$ for $a \le x \le b$ and $c \le y \le d$.

For details regarding `Plot3D` and its options enter `?Plot3D` or `??Plot3D`. (See Figure 1.32.) Frequently used options are illustrated in the following examples.

Graphs of several level curves of $z = f(x, y)$ are generated with

```
ContourPlot[f[x,y],{x,a,b},{y,c,d}].
```

For details regarding `ContourPlot` and its options enter `?ContourPlot` or `??ContourPlot`. (See Figure 1.33.)

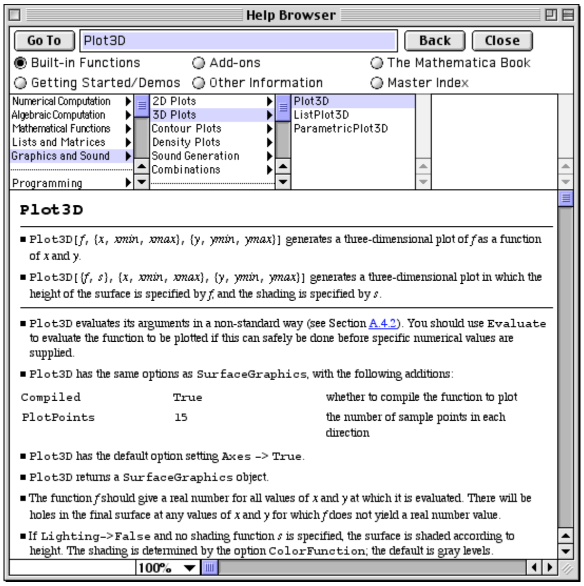


Figure 1.32: The `Plot3D` help window

Example 17. Let

$$f(x, y) = \frac{x^2 y}{x^4 + 4y^2}.$$

(a) Calculate $f(1, -1)$. (b) Graph $f(x, y)$ and several contour plots of $f(x, y)$ on a region containing $(0, 0)$.

Solution. After defining $f(x, y)$, we evaluate $f(1, -1) = -1/5$.

```
In[107]:= f[x_, y_] = x^2y/(x^4 + 4y^2)
```

$$Out[107]= \frac{x^2\ y}{x^4 + 4\ y^2}$$

```
In[108]:= f[1, -1]
```

$$Out[108]= -\frac{1}{5}$$

Next, we use `Plot3D` to graph $f(x, y)$ for $-1/2 \le x \le 1/2$ and $-1/2 \le y \le 1/2$ in Figure 1.34. We illustrate the use of the `Axes`, `Boxed`, and `PlotPoints` options.

```
In[109]:= Plot3D[f[x, y], {x, -1/2, 1/2}, {y, -1/2, 1/2},
             Axes- > Automatic, Boxed- > False, PlotPoints- > {50, 50}]
```

Two contour plots are generated with `ContourPlot`. The second illustrates the use of the `PlotPoints`, `Frame`, `ContourShading`, `Axes`, and `AxesOrigin` options. (See Figure 1.35.)

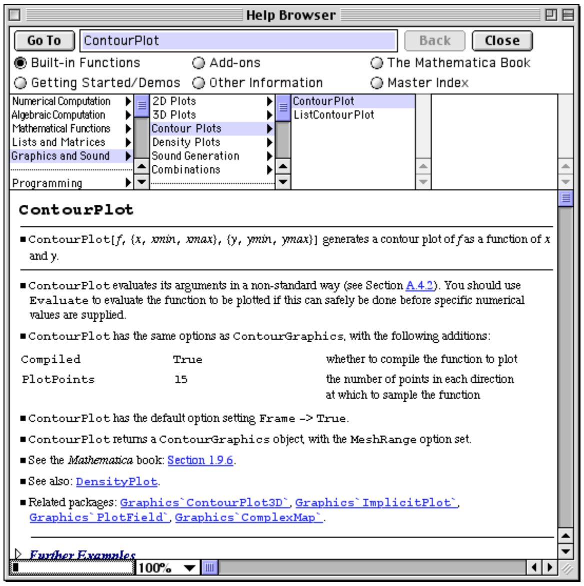


Figure 1.33: The `ContourPlot` help window

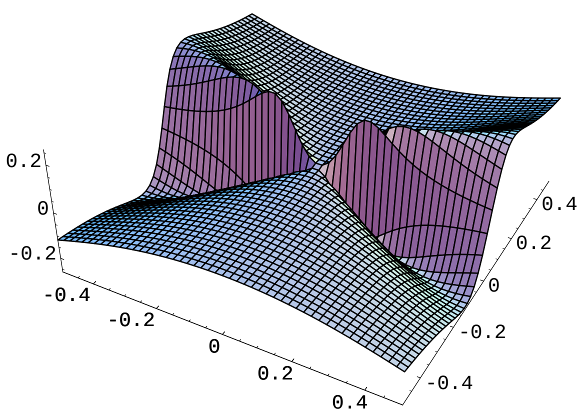


Figure 1.34: Three-dimensional plot of $f(x, y)$

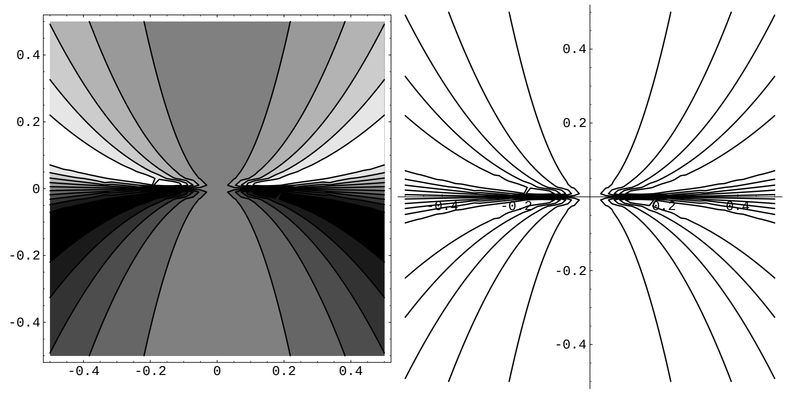


Figure 1.35: Two contour plots of $f(x, y)$

```
In[110]:= ContourPlot[f[x, y], {x, -1/2, 1/2}, {y, -1/2, 1/2},
            PlotPoints- > {50, 50}]

In[111]:= ContourPlot[f[x, y], {x, -1/2, 1/2},
            {y, -1/2, 1/2}, PlotPoints- > {60, 60}, Frame- > False,
            ContourShading- > False, Axes- > Automatic,
            AxesOrigin- > {0, 0}]
```

□

`ContourPlot` is especially useful when graphing equations. The graph of the equation $f(x, y) = C$, where C is a constant, is the same as the contour plot of $z = f(x, y)$ corresponding to C. That is, the graph of $f(x, y) = C$ is the same as the level curve of $z = f(x, y)$ corresponding to $z = C$.

Example 18. Graph the unit circle, $x^2 + y^2 = 1$.

Solution. We first graph $z = x^2 + y^2$ for $-4 \le x \le 4$ and $-4 \le y \le 4$ with `Plot3D` in Figure 1.36.

```
In[112]:= Plot3D[x^2 + y^2, {x, -4, 4}, {y, -4, 4}]
```

The graph of $x^2 + y^2 = 1$ is the graph of $z = x^2 + y^2$ corresponding to $z = 1$. We use `ContourPlot` together with the `Contours` option to graph this equation in Figure 1.37.

```
In[113]:= ContourPlot[x^2 + y^2, {x, -3/2, 3/2}, {y, -3/2, 3/2},
            Contours- > {1}, ContourShading- > False]
```

Multiple graphs can be generated as well. As an illustration, we graph $x^2 + y^2 = C$ for $C = 1$, 4, and 9 in Figure 1.38.

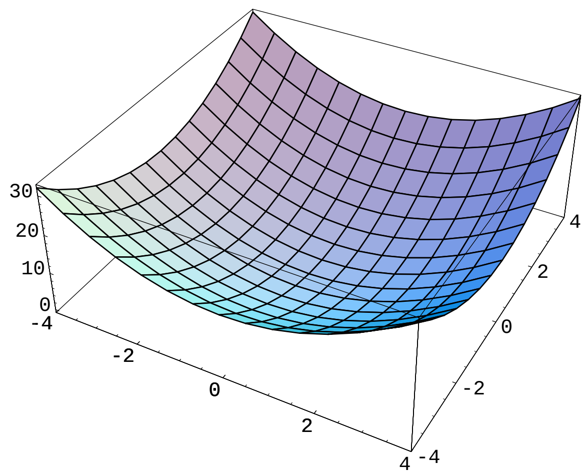


Figure 1.36: Three-dimensional plot of $z = x^2 + y^2$

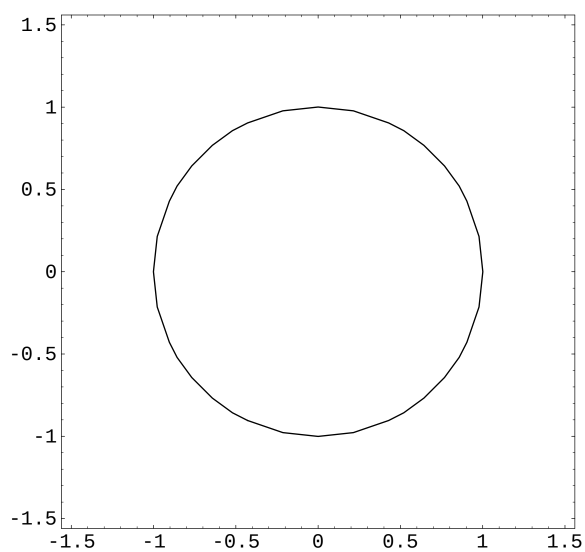


Figure 1.37: The unit circle, $x^2 + y^2 = 1$

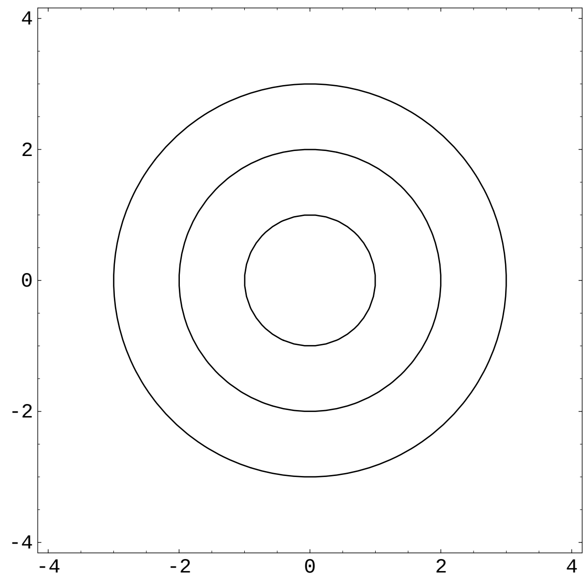


Figure 1.38: Graphs of $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, and $x^2 + y^2 = 9$

```
In[114]:= ContourPlot[x^2 + y^2, {x, -4, 4}, {y, -4, 4},
            Contours- > {1, 4, 9}, ContourShading- > False,
            PlotPoints- > {50, 50}]
```

□

We can use `ParametricPlot3D` to generate graphs of surfaces defined parametrically.

Example 19. A parametrization of **Umbilic Torus NC** is given by $\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}$, $-\pi \le s \le \pi$, $-\pi \le t \le \pi$, where

$$x = \left[7 + \cos\left(\frac{1}{3}s - 2t\right) + 2\cos\left(\frac{1}{3}s + t\right)\right]\sin s$$
$$y = \left[7 + \cos\left(\frac{1}{3}s - 2t\right) + 2\cos\left(\frac{1}{3}s + t\right)\right]\cos s$$

and

$$z = \sin\left(\frac{1}{3}s - 2t\right) + 2\sin\left(\frac{1}{3}s + t\right).$$

Graph the torus.

Solution. We define x, y, and z.

```
In[115]:= x[s_, t_] = (7 + Cos[1/3s - 2t] + 2Cos[1/3s + t]) Sin[s];
          y[s_, t_] = (7 + Cos[1/3s - 2t] + 2Cos[1/3s + t]) Cos[s];
          z[s_, t_] = Sin[1/3s - 2t] + 2Sin[1/3s + t];
```

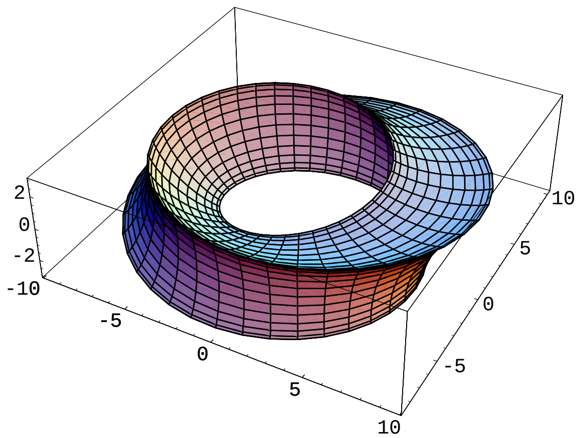


Figure 1.39: Umbilic torus

The torus is then graphed with `ParametricPlot3D` in Figure 1.39. We illustrate the use of the `PlotPoints` option.

```
In[116]:= ParametricPlot3D[{x[s, t], y[s, t], z[s, t]},
            {s, -π, π}, {t, -π, π}, PlotPoints- > {40, 40}]
```

□

1.6 Exercises

1. Solve

$$x - 2y - 4 = 0$$
$$6x + 2y - 10 = 0$$

and confirm your result graphically.

2. (a) Graph $y = x$ and $y = x^2$. (b) Find the points at which the graphs intersect.

3. Let $f(x) = 2x - x^2$. (a) Graph $f(x)$ for $-1 \le x \le 3$ to scale. (b) Compute and simplify $\frac{f(x+h)-f(x)}{h}$.

4. (a) Graph $y = \sin x$ and $y = \cos x$ to scale for $0 \le x \le 2\pi$. (b) Find all intersection points *exactly*.

5. (a) Find the points where the graphs of $y = x - 1$ and $y = x^2 - x - 1$ intersect. (b) Graph the equations together to confirm your result.

6. Solve

$$x + y = 0$$
$$x^3 - 5x - y = 0$$

and confirm your result graphically.

7. Graph $x = y^3 - 4y^2 + 3y$ and $x = y^2 - y$ together. Locate all x and y-intercepts and intersection points.

Chapter 2

Calculus

Chapter 2 introduces Mathematica's calculus commands. The examples used to illustrate the various commands are similar to examples routinely done in three semester calculus courses.

2.1 Limits

One of the first topics discussed in calculus is that of limits. Mathematica can be used to investigate limits graphically and numerically. In addition, Mathematica uses the command

```
Limit[f[x],x->a]
```

to find $\lim_{x\to a} f(x)$, the limit of $f(x)$ as x approaches the value a, where a can be a finite number, positive infinity (∞), or negative infinity ($-\infty$). Mathematica uses `Infinity` to represent ∞.

Remark. To define a function of a single variable, $f(x) = \textit{expression in } x$, enter `f[x_]=expression in x`. To generate a basic plot of $y = f(x)$ for $a \le x \le b$, enter `Plot[f[x],{x,a,b}]`.

Example 20. Use a graph and table of values to investigate $\lim_{x\to 0} \frac{\sin 3x}{x}$.

Solution. We clear all prior definitions of f, define $f(x) = \frac{\sin 3x}{x}$, and then graph $f(x)$ on the interval $[-\pi, \pi]$ with `Plot`.

`Clear[f]` clears all prior definitions of f, if any. Clearing function definitions before defining new ones helps eliminate any possible confusion and/or ambiguities.

```
In[117]:= Clear[f]

          f[x_] = Sin[3x]/x;
          Plot[f[x], {x, -π, π}]
```

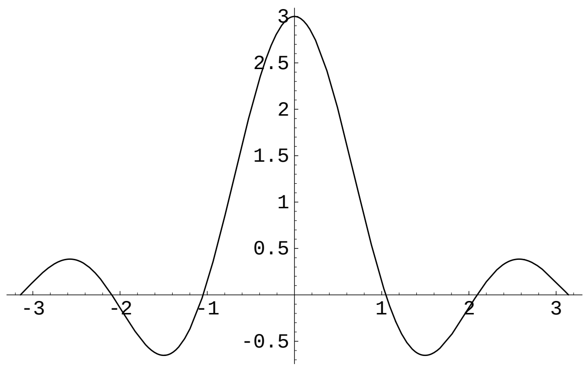


Figure 2.1: Graph of $f(x) = \frac{\sin 3x}{x}$ on the interval $[-\pi, \pi]$.

From the graph shown in Figure 2.1, we might, correctly, conclude that $\lim_{x\to 0} \frac{\sin 3x}{x} = 3$. Further evidence that $\lim_{x\to 0} \frac{\sin 3x}{x} = 3$ can be obtained by computing the values of $f(x)$ for values of x "near" 0. In the following, we use `Random` to define `xvals` to be a table of 6 "random" real numbers. The first number in `xvals` is between -1 and 1, the second between $-1/10$ and $1/10$, and so on.

`Random[Real,{a,b}]` returns a "random" real number between a and b. Because we are generating "random" numbers, your results will differ from those obtained here.

```
In[118]:= xvals = Table[Random[Real, {-1/10^n, 1/10^n}], {n, 0, 5}]
Out[118]= {0.244954, 0.0267254,
           0.00433248, -0.000864136,
           -0.0000995605, 1.83354 10^-6}
```

`Map[f,{x1,x2,x3,...,xn}]` returns the set $\{f(x_1), f(x_2), ..., f(x_n)\}$.

We then use `Map` to compute the value of $f(x)$ for each x in `xvals`.

```
In[119]:= Map[f, xvals]
Out[119]= {2.73719, 2.99679,
           2.99992, 3.,
           3., 3.}
```

From these values, we might again correctly deduce that $\lim_{x\to 0} \frac{\sin 3x}{x} = 3$. Of course, these results do not prove that $\lim_{x\to 0} \frac{\sin 3x}{x} = 3$ but they are helpful in convincing us that $\lim_{x\to 0} \frac{\sin 3x}{x} = 3$. □

2.1.1 Computing Limits

Some limits involving rational functions can be computed by factoring the numerator and denominator.

Example 21. Compute

$$\lim_{x\to -9/2} \frac{2x^2 + 25x + 72}{72 - 47x - 14x^2}.$$

Solution. We define `frac1` to be the rational expression $\frac{2x^2+25x+72}{72-47x-14x^2}$. We then attempt to compute the value of $\frac{2x^2+25x+72}{72-47x-14x^2}$ if $x = -9/2$ by using `ReplaceAll(/.)` to evaluate `frac1` if $x = -9/2$ but see that it is undefined.

```
In[120]:= frac1 = (2x^2 + 25x + 72)/(72 - 47x - 14x^2);
          frac1/.x- > -9/2
Power :: "infy" :
  "Infiniteexpression10 encountered."
∞ :: "indet" :  "Indeterminateexpression0

InterpretationBox[ComplexInfinity,
     DirectedInfinity[]] encountered."

Out[120]= Indeterminate
```

Factoring the numerator and denominator with `Factor`, `Numerator`, and `Denominator`, we see that

$$\lim_{x\to -9/2} \frac{2x^2+25x+72}{72-47x-14x^2} = \lim_{x\to -9/2} \frac{(x+8)(2x+9)}{(8-7x)(2x+9)} = \lim_{x\to -9/2} \frac{x+8}{8-7x}.$$

The fraction $(x+8)/(8-7x)$ is named `frac2` and the limit is evaluated by computing the value of `frac2` if $x = -9/2$.

```
In[121]:= Factor[Numerator[frac1]]
Out[121]= (8 + x) (9 + 2 x)

In[122]:= Factor[Denominator[frac1]]
Out[122]= -(9 + 2 x) (-8 + 7 x)

In[123]:= frac2 = Simplify[frac1]
            8 + x
Out[123]= -------
          8 - 7 x
In[124]:= frac2/.x- > -9/2
           7
Out[124]= --
          79
```

`Simplify[expression]` attempts to simplify *expression*.

We conclude that

$$\lim_{x\to -9/2} \frac{2x^2+25x+72}{72-47x-14x^2} = \frac{7}{79}.$$

□

We can also use the `Limit` command to evaluate frequently encountered limits.

`Limit[f[x],x->a]`

attempts to compute $\lim_{x\to a} f(x)$.

Thus, entering

```
In[125]:= Limit[(2x^2 + 25x + 72)/(72 - 47x - 14x^2), x- > -9/2]
           7
Out[125]= --
          79
```

computes $\lim_{x\to -9/2} \frac{2x^2+25x+72}{72-47x-14x^2} = 7/79$.

Example 22. Calculate each limit: (a) $\lim_{x\to -5/3} \frac{3x^2-7x-20}{21x^2+14x-35}$; (b) $\lim_{x\to 0} \frac{\sin x}{x}$; (c) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$; (d) $\lim_{x\to 0} \frac{e^{3x}-1}{x}$; (e) $\lim_{x\to\infty} e^{-2x}\sqrt{x}$; and (f) $\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

Solution. In each case, we use `Limit` to evaluate the indicated limit. Entering

```
In[126]:= Limit[(3x^2 - 7x - 20)/(21x^2 + 14x - 35), x-> -5/3]
Out[126]= 17/56
```

computes

$$\lim_{x\to -5/3} \frac{3x^2-7x-20}{21x^2+14x-35} = \frac{17}{56};$$

and entering

```
In[127]:= Limit[Sin[x]/x, x-> 0]
Out[127]= 1
```

computes

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Mathematica represents ∞ by `Infinity`. Thus, entering

```
In[128]:= Limit[(1 + 1/x)^x, x-> ∞]
Out[128]= e
```

computes

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e.$$

Entering

```
In[129]:= Limit[(Exp[3x] - 1)/x, x-> 0]
Out[129]= 3
```

computes

$$\lim_{x\to 0} \frac{e^{3x}-1}{x} = 3.$$

Entering

```
In[130]:= Limit[Exp[-2x]Sqrt[x], x-> ∞]
Out[130]= 0
```

computes $\lim_{x\to\infty} e^{-2x}\sqrt{x} = 0$, and entering

Because $\ln x$ is undefined for $x \le 0$, a right-hand limit is mathematically necessary, even though Mathematica's `Limit` function computes the limit correctly without the distinction.

```
In[131]:= Limit[1/Log[x] - 1/(x - 1), x-> 1]
Out[131]= 1/2
```

computes

$$\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \frac{1}{2}.$$

□

We can often use the `Limit` command to compute symbolic limits.

Example 23. If \$$P$ is compounded n times per year at an annual interest rate of r, the value of the account, A, after t years is given by

$$A = \left(1 + \frac{r}{n}\right)^{nt}.$$

The formula for continuously compounded interest is obtained by taking the limit of this expression as $t \to \infty$.

Solution. The formula for continuously compounded interest, $A = Pe^{rt}$, is obtained using `Limit`.

```
In[132]:= Limit[p(1 + r/n)^(n t), n-> ∞]
Out[132]= e^(r t) p
```

□

2.1.2 One-Sided Limits

In some cases, Mathematica can compute certain one sided limits . The command

```
Limit[f[x],x->a,Direction->1]
```

attempts to compute $\lim_{x\to a^-} f(x)$ while

```
Limit[f[x],x->a,Direction->-1]
```

attempts to compute $\lim_{x\to a^+} f(x)$.

Example 24. Compute (a) $\lim_{x\to 0^+} |x|/x$; (b) $\lim_{x\to 0^-} |x|/x$; (c) $\lim_{x\to 0^+} e^{-1/x}$; and (d) $\lim_{x\to 0^-} e^{-1/x}$.

Solution. Even though $\lim_{x\to 0} |x|/x$ does not exist, $\lim_{x\to 0^+} |x|/x = 1$ and $\lim_{x\to 0^-} |x|/x = -1$, as we see using `Limit` together with the `Direction->1` and `Direction->-1` options, respectively.

```
In[133]:= Limit[Abs[x]/x, x-> 0]
Out[133]= 1

In[134]:= Limit[Abs[x]/x, x-> 0, Direction-> -1]
Out[134]= 1

In[135]:= Limit[Abs[x]/x, x-> 0, Direction-> 1]
Out[135]= -1
```

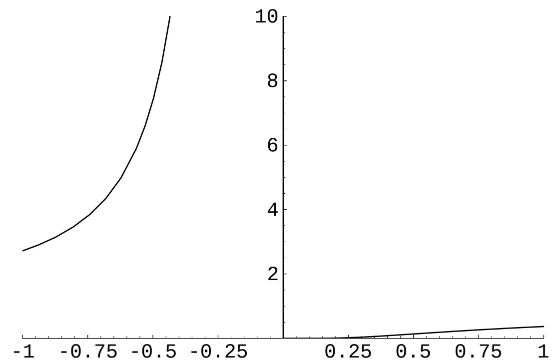


Figure 2.2: Graph of $y = e^{-1/x}$ on the interval $[-3/2, 3/2]$.

The `Direction->-1` and `Direction->1` options are used to calculate the correct values for (c) and (d), respectively. For (c), we have:

```
In[136]:= Limit[1/x, x- > 0]
Out[136]= ∞

In[137]:= Limit[1/x, x- > 0, Direction- > -1]
Out[137]= ∞

In[138]:= Limit[1/x, x- > 0, Direction- > 1]
Out[138]= -∞

In[139]:= Limit[Exp[-1/x], x- > 0]
Out[139]= 0
```

Similarly, for (d) we have:

```
In[140]:= Limit[Exp[-1/x], x- > 0, Direction- > 1]
Out[140]= ∞

In[141]:= Limit[Exp[-1/x], x- > 0, Direction- > -1]
Out[141]= 0
```

We confirm these results by graphing $y = e^{-1/x}$ with `Plot` in Figure 2.2.

```
In[142]:= Plot[Exp[-1/x], {x, -3/2, 3/2},
            PlotRange- > {{-1, 1}, {0, 10}}]
```

□

The `Limit` command together with the `Direction->1` and `Direction->-1` options is a ”fragile” command and should be used with caution because its results are unpredictable, especially for the beginner. It is wise to check or confirm results using a different technique for nearly all problems faced by the beginner.

2.2 Differential Calculus

2.2.1 Definition of the Derivative

The **derivative** of $y = f(x)$ is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \tag{2.1}$$

provided this limit exists.

The `Limit` command can be used along with `Simplify` to compute the derivative of a function using the definition of the derivative.

Remark. To define a function of a single variable, $f(x) = \textit{expression in } x$, enter `f[x_]=expression in x`. To generate a basic plot of $y = f(x)$ for $a \leq x \leq b$, enter `Plot[f[x],{x,a,b}]`.

Example 25. Use the definition of the derivative to compute the derivative of (a) $f(x) = x + 1/x$, (b) $g(x) = \sqrt{x} + 1/\sqrt{x}$ and (c) $h(x) = \sin 2x$.

Solution. For (a) and (b), we first define f and g, compute the difference quotient, $(f(x+h) - f(x))/h$, simplify the difference quotient with `Simplify`, and use `Limit` to calculate the derivative.

```
In[143]:= f[x_] = x + 1/x;
          s1 = (f[x + h] - f[x])/h
```

Out[143]= $\dfrac{h - \frac{1}{x} + \frac{1}{h+x}}{h}$

```
In[144]:= s2 = Simplify[s1]
```

Out[144]= $\dfrac{-1 + h\ x + x^2}{x\ (h + x)}$

```
In[145]:= Limit[s2, h- > 0]
```

Out[145]= $\dfrac{-1 + x^2}{x^2}$

```
In[146]:= g[x_] = 1/Sqrt[x]
          s1 = (g[x + h] - g[x])/h
```

Out[146]= $\dfrac{1}{\sqrt{x}}$

Out[146]= $\dfrac{-\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{h+x}}}{h}$

```
In[147]:= s2 = Together[s1]
```

Out[147]= $\dfrac{\sqrt{x} - \sqrt{h + x}}{h\ \sqrt{x}\ \sqrt{h + x}}$

```
In[148]:= Limit[s2, h- > 0]
```

Out[148]= $-\dfrac{1}{2\ x^{3/2}}$

For (c), we define h and then use `TrigExpand` to simplify the difference quotient. We use `Limit` to compute the derivative. The result indicates that $\frac{d}{dx}(\sin 2x) = 2\cos 2x$.

```
In[149]:= h[x_] = Sin[2x];
          s2 = (h[x + h] - h[x])/h
Out[149]= (-Sin[2 x] + Sin[2 (h + x)])/h
In[150]:= s2 = TrigExpand[s2]
Out[150]= 1/h (2 Cos[h] Cos[x]^2 Sin[h] - 2 Cos[x] Sin[x] +
            2 Cos[h]^2 Cos[x] Sin[x] - 2 Cos[x] Sin[h]^2 Sin[x] -
            2 Cos[h] Sin[h] Sin[x]^2)
In[151]:= s3 = Limit[s2, h- > 0]
Out[151]= 2 Cos[2 x]
```

□

If the derivative of $y = f(x)$ exists at $x = a$, a geometric interpretation of $f'(a)$ is that $f'(a)$ is the slope of the line tangent to the graph of $y = f(x)$ at the point $(a, f(a))$.

To motivate the definition of the derivative, many calculus texts choose a value of x, $x = a$, and then draw the graph of the secant line passing through the points $(a, f(a))$ and $(a + h, f(a + h))$ for "small" values of h to show that as h approaches 0, the secant line approaches the tangent line. An equation of the secant line passing through the points $(a, f(a))$ and $(a + h, f(a + h))$ is given by

$$y - f(a) = \frac{f(a+h) - f(a)}{(a+h) - a}(x - a) \quad \text{or} \quad y = \frac{f(a+h) - f(a)}{h}(x - a) + f(a).$$

Example 26. If $f(x) = 9 - 4x^2$, graph $f(x)$ together with the secant line containing $(1, f(1))$ and $(1 + h, f(1 + h))$ for various values of h.

Solution. We define $f(x) = 9 - 4x^2$ and $y(x, h)$ to be a function returning the line containing $(1, f(1))$ and $(1 + h, f(1 + h))$.

```
In[152]:= f[x_] = 9 - 4x^2;
          y[x_, h_] = (f[1 + h] - f[1])/h(x - 1) + f[1];
```

In the following, we use `Do` to show the graphs of $f(x)$ and $y(x, h)$ for $h = 1, 2, \ldots, 9$. The resulting animation can be played and controlled from the Mathematica menu. (See Figure 2.3.)

```
In[153]:= Do[Plot[{f[x], y[x, 1/h]},
              {x, -3, 3}, PlotRange- > {-10, 10}],
             {h, 1, 10}]
```

If instead the command is entered as

```
In[154]:= listofgraphics = Table[Plot[{f[x], y[x, 1/h]}, {x, -3, 3},
                PlotRange- > {-10, 10}, DisplayFunction- > Identity],
              {h, 1, 10}]
In[155]:= toshow = Partition[listofgraphics, 3]
In[156]:= Show[GraphicsArray[toshow]]
```

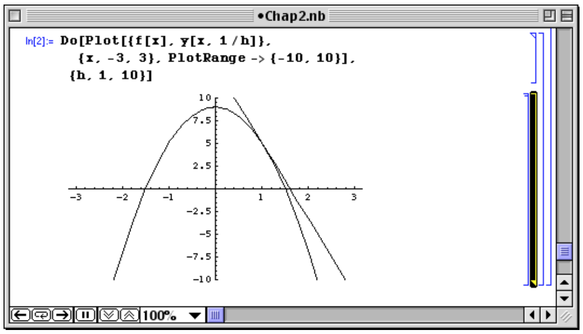


Figure 2.3: An animation

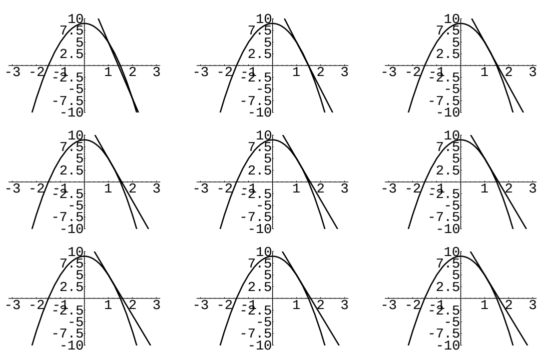


Figure 2.4: A graphics array

the result is displayed as a graphics array. (See Figure 2.4.) □

The functions `D` and `'` are used to differentiate functions. Assuming that $y = f(x)$ is differentiable,

1. `D[f[x],x]` computes and returns $f'(x) = df/dx$,
2. `f'[x]` computes and returns $f'(x) = df/dx$,
3. `f''[x]` computes and returns $f^{(2)}(x) = d^2f/dx^2$, and
4. `D[f[x],{x,n}]` computes and returns $f^{(n)}(x) = d^nf/dx^n$.

Mathematica knows the numerous differentiation rules, including the product, quotient, and chain rules. Thus, entering

```
In[157]:= Clear[f, g]

          D[f[x]g[x], x]
Out[157]= g[x] f′[x] + f[x] g′[x]
```

shows us that $\frac{d}{dx}(f(x) \cdot g(x) = f'(x)g(x) + f(x)g'(x)$; entering

```
In[158]:= Together[D[f[x]/g[x], x]]
           g[x] f′[x] - f[x] g′[x]
Out[158]= -------------------------
                   g[x]^2
```

shows us that $\frac{d}{dx}(f(x)/g(x)) = (f'(x)g(x) - f(x)g'(x))/(g(x))^2$; and entering

```
In[159]:= D[f[g[x]], x]
Out[159]= f′[g[x]] g′[x]
```

shows us that $\frac{d}{dx}(f(g(x)) = f'(g(x))\,g'(x)$.

Example 27. Compute the first and second derivatives of (a) $y = x^4 + \frac{4}{3}x^3 - 3x^2$, (b) $f(x) = 4x^5 - \frac{5}{2}x^4 - 10x^3$, (c) $y = \sqrt{e^{2x} + e^{-2x}}$, and (d) $y = (1 + 1/x)^x$.

Solution. For (a), we use `D`.

```
In[160]:= D[x^4 + 4/3x^3 - 3x^2, x]
Out[160]= -6 x + 4 x^2 + 4 x^3

In[161]:= D[x^4 + 4/3x^3 - 3x^2, {x, 2}]
Out[161]= -6 + 8 x + 12 x^2
```

For (b), we first define f and then use `'` together with `Factor` to calculate and factor $f'(x)$ and $f''(x)$.

```
In[162]:= f[x_] = 4x^5 - 5/2x^4 - 10x^3;
          Factor[f′[x]]

          Factor[f′′[x]]
```

```
Out[162]= 10 x^2 (1 + x) (-3 + 2 x)
Out[162]= 10 x (-6 - 3 x + 8 x^2)
```

For (c), we use `simplify` together with `D` to calculate and simplify y' and y''.

```
In[163]:= D[Sqrt[Exp[2x] + Exp[-2x]], x]
```

$$\text{Out[163]= } \frac{-2\, e^{-2x} + 2\, e^{2x}}{2\sqrt{e^{-2x} + e^{2x}}}$$

```
In[164]:= D[Sqrt[Exp[2x] + Exp[-2x]], {x, 2}]//Simplify
```

$$\text{Out[164]= } \frac{\sqrt{e^{-2x} + e^{2x}}\,(1 + 6\, e^{4x} + e^{8x})}{(1 + e^{4x})^2}$$

By hand, (d) would require logarithmic differentiation. The second derivative would be particularly difficult to compute by hand. Mathematica quickly computes and simplifies each derivative.

```
In[165]:= Simplify[D[(1 + 1/x)^x, x]]
```

$$\text{Out[165]= } \frac{\left(1 + \frac{1}{x}\right)^x \left(-1 + (1 + x)\ \mathrm{Log}\left[1 + \frac{1}{x}\right]\right)}{1 + x}$$

```
In[166]:= Simplify[D[(1 + 1/x)^x, {x, 2}]]
```

$$\text{Out[166]= } \frac{\left(1 + \frac{1}{x}\right)^x \left(-1 + x - 2\, x\, (1 + x)\ \mathrm{Log}\left[1 + \frac{1}{x}\right] + x\, (1 + x)^2\ \mathrm{Log}\left[1 + \frac{1}{x}\right]^2\right)}{x\, (1 + x)^2}$$

□

The command `Map[f, list]` applies the function `f` to each element of the list `list`. Thus, if you are computing the derivatives of a large number of functions, you can use `Map` together with `D`.

Remark. A built-in Mathematica function is **threadable** if `f[list]` returns the same result as `Map[f, list]`. Many familiar functions like `D` and `Integrate` are threadable.

Example 28. Compute the first and second derivatives of $\sin x$, $\cos x$, $\tan x$, $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$.

Solution. Notice that lists are contained in braces. Thus, entering

```
In[167]:= Map[D[#, x]&,
            {Sin[x], Cos[x], Tan[x], ArcSin[x], ArcCos[x], ArcTan[x]}]
```

$$\text{Out[167]= } \left\{\mathrm{Cos}[x], -\mathrm{Sin}[x], \mathrm{Sec}[x]^2, \frac{1}{\sqrt{1 - x^2}}, -\frac{1}{\sqrt{1 - x^2}}, \frac{1}{1 + x^2}\right\}$$

computes the first derivative of the three trigonometric functions and their inverses while entering

```
In[168]:= Map[D[#, {x, 2}]&,
            {Sin[x], Cos[x], Tan[x], ArcSin[x], ArcCos[x], ArcTan[x]}]
```

$$\text{Out[168]= } \left\{-\mathrm{Sin}[x], -\mathrm{Cos}[x], 2\ \mathrm{Sec}[x]^2\ \mathrm{Tan}[x], \frac{x}{(1 - x^2)^{3/2}}, -\frac{x}{(1 - x^2)^{3/2}}, -\frac{2\, x}{(1 + x^2)^2}\right\}$$

computes the second derivative of the three trigonometric functions and their inverses. Because D is threadable, the same results are obtained with

```
In[169]:= D[
           {Sin[x], Cos[x], Tan[x], ArcSin[x], ArcCos[x], ArcTan[x]},
           x]
```

Out[169]= $\left\{\text{Cos}[x], -\text{Sin}[x], \text{Sec}[x]^2, \frac{1}{\sqrt{1-x^2}}, -\frac{1}{\sqrt{1-x^2}}, \frac{1}{1+x^2}\right\}$

```
In[170]:= D[
           {Sin[x], Cos[x], Tan[x], ArcSin[x], ArcCos[x], ArcTan[x]},
           {x, 2}]
```

Out[170]= $\left\{-\text{Sin}[x], -\text{Cos}[x], 2\ \text{Sec}[x]^2\ \text{Tan}[x], \frac{x}{(1-x^2)^{3/2}}, -\frac{x}{(1-x^2)^{3/2}}, -\frac{2\ x}{(1+x^2)^2}\right\}$

□

Implicit Differentiation

If an equation contains two variables, x and y, implicit differentiation can be carried out by explicitly declaring y to be a function of x, $y = y(x)$, and using D or by using the Dt command.

Example 29. Find $y' = dy/dx$ if (a) $\cos(e^{xy}) = x$ and (b) $\ln(x/y) + 5xy = 3y$.

Solution. For (a) we illustrate the use of D. Notice that we are careful to specifically indicate that $y = y(x)$. First we differentiate with respect to x

```
In[171]:= s1 = D[Cos[Exp[x y[x]]] - x, x]
Out[171]= BoxData(-1 - e^(x y[x]) Sin[e^(x y[x])] (y[x] + x y'[x]))
```

and then we solve the resulting equation for $y' = dy/dx$ with Solve.

```
In[172]:= Solve[s1 == 0, y'[x]]
```

Out[172]= BoxData$\left(\left\{\left\{y'[x] \to -\frac{e^{-x\ y[x]}\ \text{Csc}[e^{x\ y[x]}]\ (1+e^{x\ y[x]}\ \text{Sin}[e^{x\ y[x]}]\ y[x])}{x}\right\}\right\}\right)$

For (b), we use Dt. When using Dt, we interpret Dt[x]= 1 and Dt[y]= $y' = dy/dx$. Thus, entering

```
In[173]:= s2 = Dt[Log[x/y] + 5x y - 3y]
```

Out[173]= $5\ y\ \text{Dt}[x] - 3\ \text{Dt}[y] + 5\ x\ \text{Dt}[y] + \frac{y\left(\frac{\text{Dt}[x]}{y} - \frac{x\ \text{Dt}[y]}{y^2}\right)}{x}$

```
In[174]:= s3 = s2/.{Dt[x]- > 1, Dt[y]- > dydx}
```

Out[174]= $-3\ \text{dydx} + 5\ \text{dydx}\ x + 5\ y + \frac{\left(-\frac{\text{dydx}\ x}{y^2} + \frac{1}{y}\right)\ y}{x}$

```
In[175]:= Solve[s3 == 0, dydx]
```

Out[175]= $\left\{\left\{\text{dydx} \to -\frac{y\ (1+5\ x\ y)}{x\ (-1-3\ y+5\ x\ y)}\right\}\right\}$

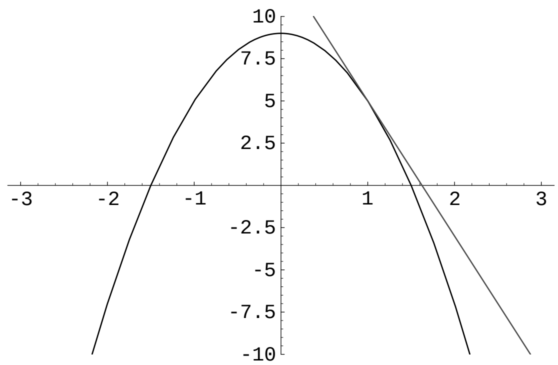


Figure 2.5: $f(x)$ together with its tangent at $(1, f(1))$

shows us that if $\ln(x/y) + 5xy = 3y$,

$$y' = \frac{dy}{dx} = -\frac{(1+5xy)y}{(5xy-3y-1)x}$$

□

2.2.2 Tangent Lines

If $f'(a)$ exists, we interpret $f'(a)$ to be the slope of the line tangent to the graph of $y = f(x)$ at the point $(a, f(a))$. An equation of the tangent is given by

$$y - f(a) = f'(a)(x-a) \quad \text{or} \quad y = f'(a)(x-a) + f(a)$$

Example 30. Find an equation of the line tangent to the graph of $f(x) = 9 - 4x^2$ at the point $(1, f(1))$.

Solution. After defining f, we see that $f(1) = 5$ and $f'(1) = -8$

```
In[176]:= f[x_] = 9 - 4x^2;
          f[1]

          f'[1]
Out[176]= 5
Out[176]= -8
```

so an equation of the line tangent to $y = f(x)$ at the point $(1, 5)$ is $y - 5 = -8(x-1)$ or $y = -8x + 13$. We can visualize the tangent at $(1, f(1))$ with `Plot`. (See Figure 2.5.)

```
In[177]:= Plot[{f[x], f'[1](x - 1) + f[1]}, {x, -3, 3},
            PlotStyle- > {GrayLevel[0], GrayLevel[0.3]},
            PlotRange- > {-10, 10}]
```

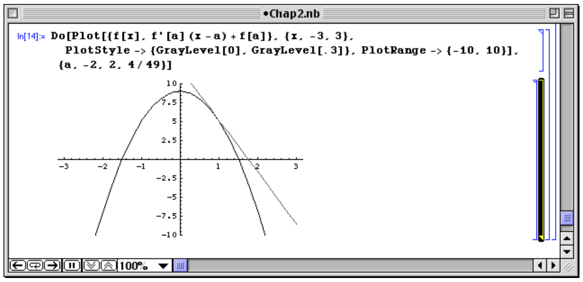


Figure 2.6: An animation

In addition, we can view a sequence of lines tangent to the graph of a function for a sequence of x values using `Do`. In the following, we use `Do` to generate graphs of $y = f(x)$ and $y = f'(a)(x - a) + f(a)$ for fifty equally spaced values of a between -3 and 3. (See Figure 2.6.)

```
In[178]:= Do[Plot[{f[x], f'[a](x - a) + f[a]}, {x, -3, 3}, PlotStyle- >
               {GrayLevel[0], GrayLevel[0.3]}, PlotRange- > {-10, 10}],
           {a, -2, 2, 4/49}]
```

On the other hand,

```
In[179]:= listofgraphics = Table[
              Plot[{f[x], f'[a](x - a) + f[a]}, {x, -3, 3}, PlotStyle- >
                 {GrayLevel[0], GrayLevel[0.3]}, PlotRange- > {-10, 10},
               DisplayFunction- > Identity], {a, -2, 2, 4/8}];
          toshow = Partition[listofgraphics, 3];
          Show[GraphicsArray[toshow]]
```

graphs $y = f(x)$ and $y = f'(a)(x - a) + f(a)$ for nine equally spaced values of a between -3 and 3 and displays the result as a graphics array. (See Figure 2.7.)

In the graphs, notice that where the tangent lines have positive slope ($f'(x) > 0$), $f(x)$ is increasing while where the tangent lines have negative slope ($f'(x) < 0$), $f(x)$ is decreasing. □

Parametric Equations and Polar Coordinates

For the parametric equations $\{x = f(t), y = g(t)\}$, $t \in I$,

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$$

and

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d/dt(dy/dx)}{dx/dt}.$$

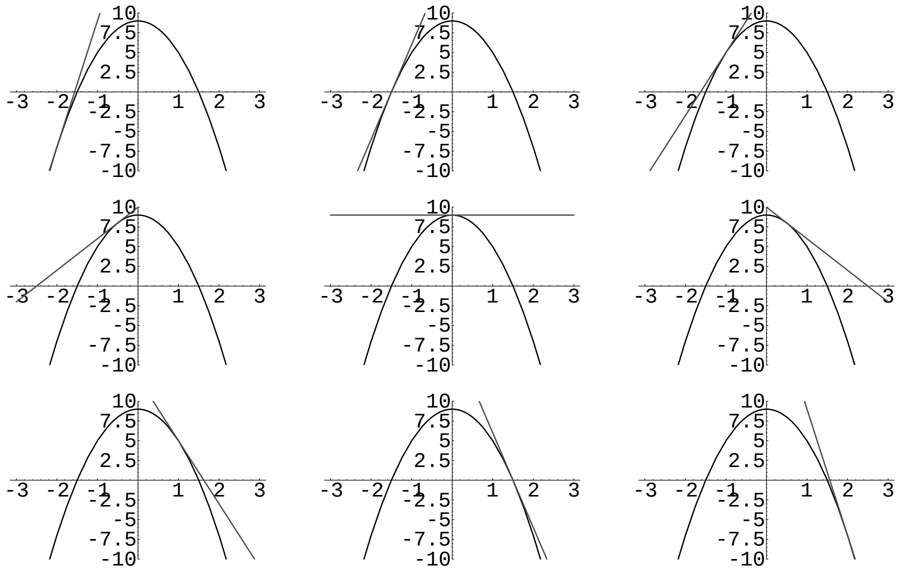


Figure 2.7: $f(x)$ together with various tangents

If $\{x = f(t), y = g(t)\}$ has a tangent line at the point $(f(a), g(a))$, parametric equations of the tangent are given by

$$x = f(a) + t f'(a) \qquad \text{and} \qquad y = g(a) + t g'(a). \tag{2.2}$$

If $g'(a) \neq 0$, we can eliminate the parameter from (2.2)

$$\frac{x - f(a)}{f'(a)} = \frac{y - g(a)}{g'(a)}$$
$$y - g(a) = \frac{g'(a)}{f'(a)}(x - f(a))$$

and obtain an equation of the tangent line in point-slope form.

```
In[180]:= l = Solve[x[a] + t x′[a] == cx, t]

          r = Solve[y[a] + t y′[a] == cy, t]
Out[180]= BoxData({{t → −(−cx + x[a])/(x′[a])}})
Out[180]= BoxData({{t → −(−cy + y[a])/(y′[a])}})
```

Example 31 (The Cycloid). The **cycloid** has parametric equations

$$x = t - \sin t \qquad \text{and} \qquad y = 1 - \cos t.$$

Graph the cycloid together with the line tangent to the graph of the cycloid at the point $(x(a), y(a))$ for various values of a between -2π and 4π.

Solution. After defining x and y we use `'` to compute dy/dt and dx/dt. We then compute $dy/dx = (dy/dt)/(dx/dt)$ and d^2y/dx^2.

```
In[181]:= x[t_] = t - Sin[t];
          y[t_] = 1 - Cos[t];
          dx = x'[t]

          dy = y'[t]

          dydx = dy/dx
Out[181]= 1 - Cos[t]
Out[181]= Sin[t]
             Sin[t]
Out[181]= ----------
          1 - Cos[t]
In[182]:= dypdt = Simplify[D[dydx, t]]
               1
Out[182]= -----------
          -1 + Cos[t]
In[183]:= secondderiv = Simplify[dypdt/dx]
                 1
Out[183]= - --------------
            (-1 + Cos[t])^2
```

We then use `ParametricPlot` to graph the cycloid for $-2\pi \le t \le 4\pi$, naming the resulting graph `p1`.

```
In[184]:= p1 = ParametricPlot[{x[t], y[t]}, {t, -2π, 4π},
              PlotStyle- > {{GrayLevel[0], Thickness[0.01]}},
              DisplayFunction- > Identity];
```

Next, we use `Table` to define `toplot` to be 40 tangent lines (2.2) using equally spaced values of a between -2π and 4π. We then graph each line `toplot` and name the resulting graph `p2`. Finally, we show `p1` and `p2` together with the `Show` function. The resulting plot is shown to scale because the lengths of the x and y-axes are equal and we include the option `AspectRatio->1`. In the graphs, notice that on intervals for which dy/dx is defined, dy/dx is a decreasing function and, consequently, $d^2y/dx^2 < 0$. (See Figure 2.8.)

```
In[185]:= toplot = Table[
              {x[a] + t x'[a], y[a] + t y'[a]}, {a, -2π, 4π, 6π/39}];
          p2 = ParametricPlot[Evaluate[toplot],
              {t, -2, 2}, PlotStyle- > GrayLevel[0.5],
              DisplayFunction- > Identity];
          Show[p1, p2, AspectRatio- > 1, PlotRange- > {-3π, 3π},
            DisplayFunction- > $DisplayFunction]
```

□

Example 32 (Orthogonal Curves). Two lines L_1 and L_2 with slopes m_1 and m_2, respectively, are **orthogonal** if their slopes are negative reciprocals: $m_1 = -1/m_2$. Extended to curves, we say that the curves C_1 and C_2 are **orthogonal** at a point of intersection if their respective tangent lines to the curves at that point are orthogonal.

Show that the family of curves with equation $x^2 + 2xy - y^2 = C$ is orthogonal to the family of curves with equation $y^2 + 2xy - x^2 = C$.

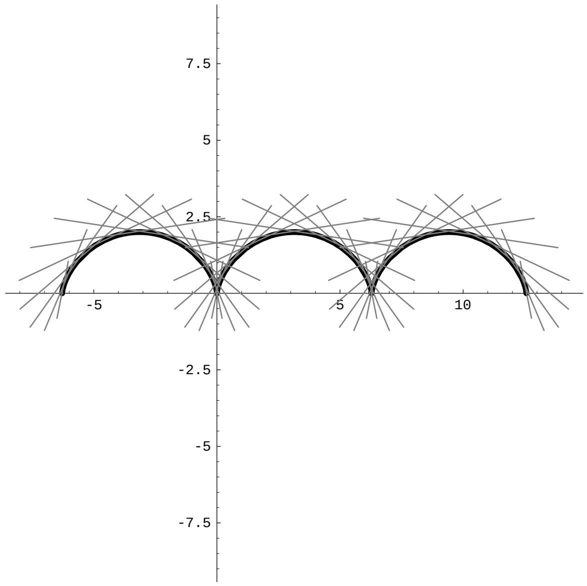


Figure 2.8: The cycloid with various tangents

Solution. We begin by defining `eq1` and `eq2` to be the left-hand sides of the equations $x^2 + 2xy - y^2 = C$ and $y^2 + 2xy - x^2 = C$, respectively.

```
In[186]:= eq1 = x^2 + 2x y - y^2;
          eq2 = y^2 + 2x y - x^2;
```

We then use `Dt` to differentiate and `Solve` to find $y' = dy/dx$. Because the derivatives are negative reciprocals, we conclude that the curves are orthogonal. We confirm this graphically by graphing several members of each family with `ContourPlot` and showing the results together. (See Figure 2.9.)

```
In[187]:= BoxData({s1 = Dt[eq1]/.{Dt[x]- > 1, Dt[y]- > dydx},  Solve[s1 ==
0, dydx]})

Out[187]= 2 x + 2 dydx x + 2 y - 2 dydx y
```

Out[187]= $\left\{\left\{\text{dydx} \to -\frac{x+y}{x-y}\right\}\right\}$

```
In[188]:= BoxData({s2 = Dt[eq2]/.{Dt[x]- > 1, Dt[y]- > dydx},  Solve[s2 ==
0, dydx]})

Out[188]= -2 x + 2 dydx x + 2 y + 2 dydx y
```

Out[188]= $\left\{\left\{\text{dydx} \to -\frac{-x+y}{x+y}\right\}\right\}$

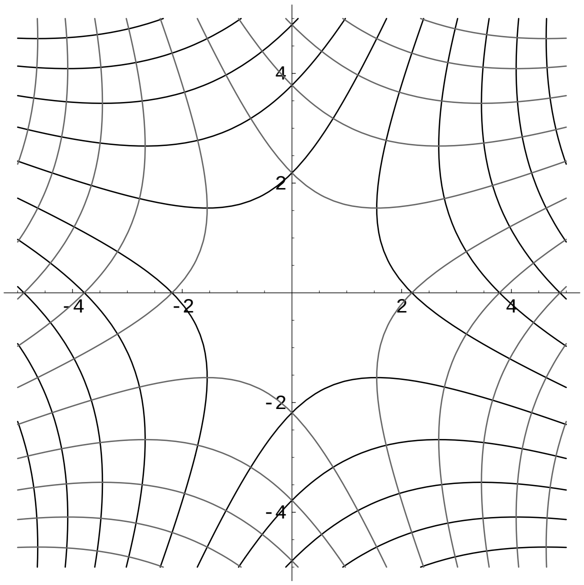


Figure 2.9: $x^2 + 2xy - y^2 = C$ and $y^2 + 2xy - x^2 = C$ for various values of C

```
In[189]:= cp1 = ContourPlot[eq1, {x, -5, 5}, {y, -5, 5},
              ContourShading- > False, ContourStyle- > GrayLevel[0],
              Frame- > False, Axes- > Automatic, AxesOrigin- > {0, 0},
              DisplayFunction- > Identity, PlotPoints- > 60];
          cp2 = ContourPlot[eq2, {x, -5, 5}, {y, -5, 5},
              ContourShading- > False, ContourStyle- > GrayLevel[0.4],
              Frame- > False, Axes- > Automatic, AxesOrigin- > {0, 0},
              DisplayFunction- > Identity, PlotPoints- > 60];
          Show[cp1, cp2, DisplayFunction- > $DisplayFunction]
```

□

2.2.3 The First Derivative Test and Second Derivative Test

Examples 30 and 31 illustrate the following properties of the first and second derivative.

Theorem 1. *Let $y = f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).*

1. *If $f'(x) = 0$ for all x in (a, b), then $f(x)$ is constant on $[a, b]$.*
2. *If $f'(x) > 0$ for all x in (a, b), then $f(x)$ is increasing on $[a, b]$.*
3. *If $f'(x) < 0$ for all x in (a, b), then $f(x)$ is decreasing on $[a, b]$.*

For the second derivative, we have the following theorem.

Theorem 2. *Let $y = f(x)$ have a second derivative on (a, b).*

1. *If $f''(x) > 0$ for all x in (a, b), then the graph of $f(x)$ is concave up on (a, b).*
2. *If $f''(x) < 0$ for all x in (a, b), then the graph of $f(x)$ is concave down on (a, b).*

The **critical points** correspond to those points on the graph of $y = f(x)$ where the tangent line is horizontal or vertical; the number $x = a$ is a **critical number** if $f'(a) = 0$ or $f'(x)$ does not exist if $x = a$. The **inflection points** correspond to those points on the graph of $y = f(x)$ where the graph of $y = f(x)$ is neither concave up nor concave down. Theorems 1 and 2 help establish the First Derivative Test and Second Derivative Test.

Theorem 3 (First Derivative Test). *Let $x = a$ be a critical number of a function $y = f(x)$ continuous on an open interval I containing $x = a$. If $f(x)$ is differentiable on I, except possibly at $x = a$, $f(a)$ can be classified as follows.*

1. *If $f'(x)$ changes from positive to negative at $x = a$, then $f(a)$ is a **relative maximum**.*
2. *If $f'(x)$ changes from negative to positive at $x = a$, then $f(a)$ is a **relative minimum**.*

Theorem 4 (Second Derivative Test). *Let $x = a$ be a critical number of a function $y = f(x)$ and suppose that $f''(x)$ exists on an open interval containing $x = a$.*

1. *If $f''(a) < 0$, then $f(a)$ is a relative maximum.*
2. *If $f''(x) > 0$, then $f(a)$ is a relative minimum.*

Example 33. Graph $f(x) = 3x^5 - 5x^3$.

Solution. We begin by defining $f(x)$ and then computing and factoring $f'(x)$ and $f''(x)$.

```
In[190]:= f[x_] = 3x^5 - 5x^3;
          d1 = Factor[f'[x]]

          d2 = Factor[f''[x]]
Out[190]= 15 (-1 + x) x^2 (1 + x)
Out[190]= 30 x (-1 + 2 x^2)
```

By inspection, we see that the critical numbers are $x = 0$, 1, and -1 while $f''(x) = 0$ if $x = 0$, $1/\sqrt{2}$, or $-1/\sqrt{2}$. Of course, these values can also be found with `Solve` as done next in `cns` and `ins`, respectively.

```
In[191]:= cns = Solve[d1 == 0]

          ins = Solve[d2 == 0]
Out[191]= {{x → -1}, {x → 0}, {x → 0}, {x → 1}}
Out[191]= {{x → 0}, {x → -1/√2}, {x → 1/√2}}
```

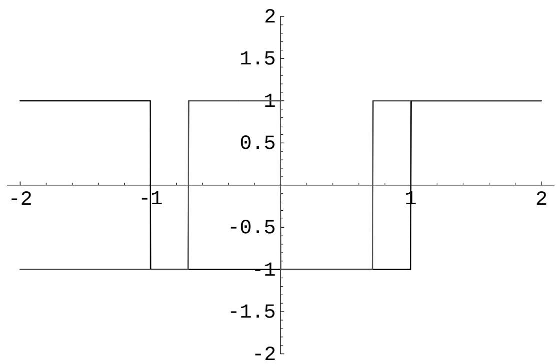


Figure 2.10: Graphs of $|f'(x)|/f'(x)$ and $|f''(x)|/f''(x)$

We find the critical and inflection points by using `/.` (`ReplaceAll`) to compute $f(x)$ for each value of x in `cns` and `ins`, respectively. The result means that the critical points are $(0,0)$, $(1,-2)$ and $(-1,2)$; the inflection points are $(0,0)$, $(1/\sqrt{2},-7\sqrt{2}/8)$, and $(-1/\sqrt{2},7\sqrt{2}/8)$. We also see that $f''(0) = 0$ so Theorem 4 cannot be used to classify $f(0)$. On the other hand, $f''(1) = 30 > 0$ and $f''(-1) = -30 < 0$ so by Theorem 4 $f(1) = -2$ is a relative minimum and $f(-1) = 2$ is a relative maximum.

```
In[192]:= cps = {x, f[x]}/.cns

          f''[x]/.cns

          ips = {x, f[x]}/.ins
Out[192]= {{-1, 2}, {0, 0}, {0, 0}, {1, -2}}
Out[192]= {-30, 0, 0, 30}
```

Out[192]= $\left\{\{0,0\},\left\{-\frac{1}{\sqrt{2}},\frac{7}{4\sqrt{2}}\right\},\left\{\frac{1}{\sqrt{2}},-\frac{7}{4\sqrt{2}}\right\}\right\}$

We can graphically determine the intervals of increase and decrease by noting that if $f'(x) > 0$ ($f'(x) < 0$), $|f'(x)|/f'(x) = 1$ ($|f'(x)|/f'(x) = -1$). Similarly, the intervals for which the graph is concave up and concave down can be determined by noting that if $f''(x) > 0$ ($f''(x) < 0$), $|f''(x)|/f''(x) = 1$ ($|f''(x)|/f''(x) = -1$). We use `Plot` to graph $|f'(x)|/f'(x)$ and $|f''(x)|/f''(x)$ in Figure 2.10.

```
In[193]:= Plot[{Abs[d1]/d1, Abs[d2]/d2}, {x, -2, 2},
            PlotStyle- > {GrayLevel[0], GrayLevel[0.3]},
            PlotRange- > {-2, 2}]
```

From the graph, we see that $f'(x) > 0$ for x in $(-\infty,-1) \cup (1,\infty)$, $f'(x) < 0$ for x in $(-1,1)$, $f''(x) > 0$ for x in $(-1/\sqrt{2},0) \cup (1/\sqrt{2},\infty)$, and $f''(x) < 0$ for x in $(-\infty,-1/\sqrt{2}) \cup (0,1/\sqrt{2})$. Thus, the graph of $f(x)$ is

- increasing and concave down for x in $(-\infty,-1)$,
- decreasing and concave down for x in $(-1,-1/\sqrt{2})$,

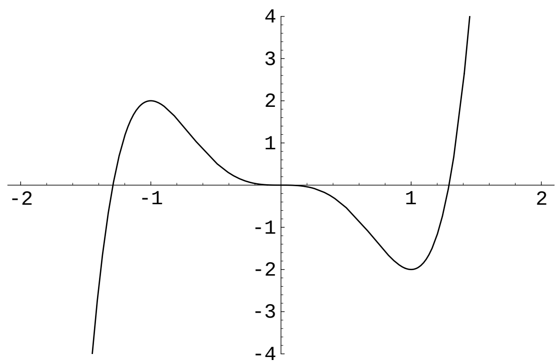


Figure 2.11: $f(x)$ for $-2 \le x \le 2$ and $-4 \le y \le 4$

- decreasing and concave up for x in $(-1/\sqrt{2}, 0)$,
- decreasing and concave down for x in $(0, 1\sqrt{2})$,
- decreasing and concave up for x in $(1/\sqrt{2}, 1)$, and
- increasing and concave up for x in $(1, \infty)$.

We also see that $f(0) = 0$ is neither a relative minimum nor maximum. To see all points of interest, our domain must contain -1 and 1 while our range must contain -2 and 2. We choose to graph $f(x)$ for $-2 \le x \le 2$; we choose the range displayed to be $-4 \le y \le 4$. (See Figure 2.11.)

```
In[194]:= Plot[f[x], {x, -2, 2}, PlotRange- > {-4, 4}]
```

□

Remember to be especially careful when working with functions that involve odd roots.

Example 34. Graph $f(x) = (x-2)^{2/3}(x+1)^{1/3}$.

Solution. We begin by defining $f(x)$ and then computing and simplifying $f'(x)$ and $f''(x)$ with `'` and `Simplify`.

```
In[195]:= f[x_] = (x - 2)^(2/3)(x + 1)^(1/3);
          d1 = Simplify[f'[x]]

          d2 = Simplify[f''[x]]
```

Out[195]= $\dfrac{x}{(-2+x)^{1/3}\,(1+x)^{2/3}}$

Out[195]= $-\dfrac{2}{(-2+x)^{4/3}\,(1+x)^{5/3}}$

By inspection, we see that the critical numbers are $x = 0$, 2, and -1. We cannot use Theorem 4 to classify $f(2)$ and $f(-1)$ because $f''(x)$ is undefined if $x = 2$ or -1. On

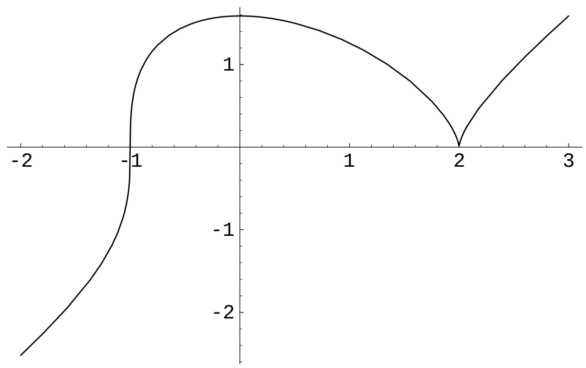


Figure 2.12: $f(x)$ for $-2 \leq x \leq 3$

the other hand, $f''(0) < 0$ so $f(0) = 2^{2/3}$ is a relative maximum. By hand, we make a sign chart to see that the graph of $f(x)$ is

- increasing and concave up on $(-\infty, -1)$,
- increasing and concave down on $(-1, 0)$,
- decreasing and concave down on $(0, 2)$, and
- increasing and concave down on $(2, \infty)$.

Hence, $f(-1) = 0$ is neither a relative minimum nor maximum while $f(2) = 0$ is a relative minimum by Theorem 3. To graph $f(x)$, we load the `RealOnly` package and then use `Plot` to graph $f(x)$ for $-2 \leq x \leq 3$ in Figure 2.12.

```
In[196]:= << Miscellaneous‘RealOnly‘

          f[0]

          Plot[f[x], {x, -2, 3}]
Out[196]= 2^(2/3)
```

□

2.2.4 Antidifferentiation

$F(x)$ is an **antiderivative** of $f(x)$ if $F'(x) = f(x)$. The symbol

$$\int f(x)\,dx$$

means ”find all antiderivatives of $f(x)$”. Because all antiderivatives of a given function differ by a constant, we usually find an antiderivative, $F(x)$, of $f(x)$ and then write

$$\int f(x)\,dx = F(x) + C,$$

where C represents an arbitrary constant. The command

```
Integrate[f[x],x]
```

attempts to compute $\int f(x)\,dx$. In the same way as `D` can differentiate many functions, `Integrate` can antidifferentiate many functions. However, antidifferentiation is a fundamentally difficult procedure so it is not difficult to find functions $f(x)$ for which the command `Integrate[f[x],x]` returns unevaluated.

Example 35. Evaluate each of the following antiderivatives: (a) $\int \frac{1}{x^2}e^{1/x}dx$, (b) $\int x^2\cos x\,dx$, (c) $\int x^2\sqrt{1+x^2}\,dx$, (d) $\int \frac{x^2-x+2}{x^3-x^2+x-1}dx$, and (e) $\int \frac{\sin x}{x}dx$.

Solution. Entering

```
In[197]:= Integrate[1/x^2 Exp[1/x], x]
Out[197]= -e^(1/x)
```

shows us that $\int \frac{1}{x^2}e^{1/x}dx = -e^{1/x} + C$. Notice that Mathematica does not automatically include the arbitrary constant, C. When computing several antiderivatives, you can use `Map` to apply `Integrate` to a list of antiderivatives. However, because `Integrate` is threadable, `Map[Integrate[#,x]&,list]` returns the same result as `Integrate[list,x]`, which we illustrate to compute (b), (c), and (d).

```
In[198]:= Integrate[{x^2 Cos[x],
              x^2 Sqrt[1 + x^2], (x^2 - x + 2)/(x^3 - x^2 + x - 1)},
            x]
```

Out[198]= $\left\{2\,x\,\mathrm{Cos}[x] - 2\,\mathrm{Sin}[x] + x^2\,\mathrm{Sin}[x],\ \sqrt{1+x^2}\left(\frac{x}{8}+\frac{x^3}{4}\right) - \frac{\mathrm{ArcSinh}[x]}{8},\ -\mathrm{ArcTan}[x] + \mathrm{Log}[-1+x]\right\}$

For (e), we see that there is not a "closed form" antiderivative of $\int \frac{\sin x}{x}dx$ and the result is given in terms of a definite integral, the **sine integral function**:

$$Si(x) = \int_0^x \frac{\sin t}{t}dt.$$

```
In[199]:= Integrate[Sin[x]/x, x]
Out[199]= SinIntegral[x]
```

□

Usually, the first antidifferentiation technique discussed is the method of u **substitution**. Suppose that $F(x)$ is an antiderivative of $f(x)$. Given

$$\int f(g(x))\,g'(x)\,dx,$$

we let $u = g(x)$ so that $du = g'(x)\,dx$. Then,

$$\int f\,(g(x))\,g'(x)\,dx = \int f(u)\,du = F(u) + C = F\,(g(x)) + C,$$

where $F(x)$ is an antiderivative of $f(x)$. After mastering u-substitutions, the **integration by parts formula**,

$$\int u\,dv = uv - \int v\,du, \tag{2.3}$$

is introduced.

Example 36. Evaluate $\int 2^x\sqrt{4^x - 1}\,dx$.

Solution. We use `Integrate` to evaluate the antiderivative. Notice that the result is *very* complicated.

```
In[200]:= Integrate[2^x Sqrt[4^x - 1], x]
```

$$\text{Out[200]=}\ \frac{2^{1+x}\ \sqrt{-1+4^x}}{2\ \text{Log}[2]+\text{Log}[4]} - \left(2^x\ \sqrt{1-4^x}\ \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{\text{Log}[2]}{\text{Log}[4]}, 1+\frac{\text{Log}[2]}{\text{Log}[4]}, 4^x\right]\ \text{Log}[4]\right)\Big/ \left(\sqrt{-1+4^x}\ \text{Log}[2]\ \ (2\ \text{Log}[2]+\text{Log}[4])\right)$$

Proceeding by hand, we let $u = 2^x$. Then, $du = 2^x\ \ln 2\,dx$ or, equivalently, $\frac{1}{\ln 2}du = 2^x\,dx$

```
In[201]:= D[2^x, x]
```

$$\text{Out[201]=}\ 2^x\ \text{Log}[2]$$

so $\int 2^x\sqrt{4^x - 1}\,dx = \frac{1}{\ln 2}\int\sqrt{u^2 - 1}\,du$. We now use `Integrate` to evaluate $\int\sqrt{u^2 - 1}\,du$

```
In[202]:= s1 = Integrate[Sqrt[u^2 - 1], u]
```

$$\text{Out[202]=}\ \frac{1}{2}\ u\ \sqrt{-1+u^2} - \frac{1}{2}\ \text{Log}\left[u+\sqrt{-1+u^2}\right]$$

and then `/.` (`ReplaceAll`)to replace u with 2^x.

```
In[203]:= s1 /.u- > 2^x
```

$$\text{Out[203]=}\ 2^{-1+x}\ \sqrt{-1+2^{2\ x}} - \frac{1}{2}\ \text{Log}\left[2^x+\sqrt{-1+2^{2\ x}}\right]$$

Clearly, proceeding by hand results in a significantly simpler antiderivative than using `Integrate` directly. □

2.3 Integral Calculus

2.3.1 Area

In integral calculus courses, the definite integral is frequently motivated by investigating the area under the graph of a positive continuous function on a closed interval. Let $y = f(x)$ be a non-negative continuous function on an interval $[a, b]$ and let n be a positive integer. If we divide $[a, b]$ into n subintervals of equal length and let $\left[x_{k-1}, x_k\right]$ denote the kth subinterval, the length of each subinterval is $(b-a)/n$ and $x_k = a + k\frac{b-a}{n}$. The area bounded by the graphs of $y = f(x)$, $x = a$, $x = b$, and the y-axis can be approximated with the sum

$$\sum_{k=1}^{n} f\left(x_k^*\right)\frac{b-a}{n}, \tag{2.4}$$

where $x_k^* \in [x_{k-1}, x_k]$. Typically, we take $x_k^* = x_{k-1} = a + (k-1)\frac{b-a}{n}$ (the left endpoint of the kth subinterval), $x_k^* = x_{k-1} = a + k\frac{b-a}{n}$ (the right endpoint of the kth subinterval), or $x_k^* = \frac{1}{2}\left(x_{k-1} + x_k\right) = a + \frac{1}{2}(2k-1)\frac{b-a}{n}$ (the midpoint of the kth subinterval). For these choices of x_k^*, (2.4) becomes

$$\frac{b-a}{n}\sum_{k=1}^{n} f\left(a + (k-1)\frac{b-a}{n}\right) \tag{2.5}$$

$$\frac{b-a}{n}\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right), \text{ and} \tag{2.6}$$

$$\frac{b-a}{n}\sum_{k=1}^{n} f\left(a + \frac{1}{2}(2k-1)\frac{b-a}{n}\right), \tag{2.7}$$

respectively. If $y = f(x)$ is increasing on $[a, b]$, (2.5) is an under approximation and (2.6) is an upper approximation: (2.5) corresponds to an approximation of the area using n inscribed rectangles; (2.6) corresponds to an approximation of the area using n circumscribed rectangles. If $y = f(x)$ is decreasing on $[a, b]$, (2.6) is an under approximation and (2.5) is an upper approximation: (2.6) corresponds to an approximation of the area using n inscribed rectangles; (2.5) corresponds to an approximation of the area using n circumscribed rectangles.

In the following example, we define the functions `leftsum[f[x],a,b,n]`, `middlesum[f[x],a,b,n]`, and `rightsum[f[x],a,b,n]` to compute (2.5), (2.7), and (2.6), respectively, and `leftbox[f[x],a,b,n]`, `middlebox[f[x],a,b,n]`, and `rightbox[f[x],a,b,n]` to generate the corresponding graphs. After you have defined these functions, you can use them with functions $y = f(x)$ that you define.

Remark. To define a function of a single variable, $f(x) =$ *expression in x*, enter `f[x_]=expression in x`. To generate a basic plot of $y = f(x)$ for $a \le x \le b$, enter `Plot[f[x],{x,a,b}]`.

Example 37. Let $f(x) = 9 - 4x^2$. Approximate the area bounded by the graph of $y =$

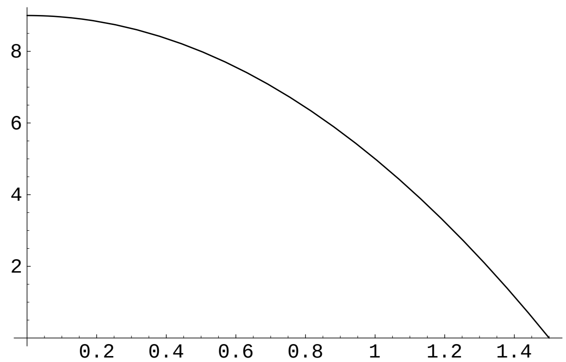


Figure 2.13: $f(x)$ for $0 \le x \le 3/2$

$f(x)$, $x = 0$, $x = 3/2$, and the y-axis using (a) 100 inscribed and (b) 100 circumscribed rectangles. (c) What is the exact value of the area?

Solution. We begin by defining and graphing $y = f(x)$ in Figure 2.13.

```
In[204]:= f[x_] = 9 - 4x^2;
          Plot[f[x], {x, 0, 3/2}]
```

The first derivative, $f'(x) = -8x$ is negative on the interval so f is decreasing on $[0, 3/2]$. Thus, an approximation of the area using 100 inscribed rectangles is given by (2.6) while an approximation of the area using 100 circumscribed rectangles is given by (2.5). After definining `leftsum`, `rightsum`, and `middlesum`, these values are computed using `leftsum` and `rightsum`. The use of `middlesum` is illustrated as well. Approximations of the sums are obtained with `N`.

`N[number]` returns a numerical approximation of *number*.

```
In[205]:= leftsum[f_, a_, b_, n_] := Module[{},
          (b - a)/n Sum[f/.x- > a + (k - 1)(b - a)/n, {k, 1, n}]];
          rightsum[f_, a_, b_, n_] := Module[{},
          (b - a)/n Sum[f/.x- > a + k(b - a)/n, {k, 1, n}]];
          middlesum[f_, a_, b_, n_] := Module[{},
          (b - a)/n Sum[f/.x- > a + 1/2(2k - 1)(b - a)/n, {k, 1, n}]];

In[206]:= l100 = leftsum[f[x], 0, 3/2, 100]

          N[l100]

          r100 = rightsum[f[x], 0, 3/2, 100]

          N[r100]

          m100 = middlesum[f[x], 0, 3/2, 100]

          N[m100]
```

Out[206]= $\frac{362691}{40000}$

```
Out[206]= 9.06728

Out[206]= 357291/40000

Out[206]= 8.93228

Out[206]= 720009/80000

Out[206]= 9.00011
```

Observe that these three values appear to be close to 9. In fact, 9 is the exact value of the area of the region bounded by $y = f(x)$, $x = 0$, $x = 3/2$, and the y-axis. To help us see why this is true, we define `leftbox`, `middlebox`, and `rightbox`, and then use these functions to visualize the situation using $n = 4$, 16, and 32 rectangles in Figure 2.14.

It is not important that you understand the syntax of these three functions at this time. Once you have entered the code, you can use them to visualize the process for your own functions, $y = f(x)$.

```
In[207]:= leftbox[f_, a_, b_, n_, opts___] :=
            Module[{z, p1, recs, ls},
          z[k_] = a + (b - a) k/n;
          p1 = Plot[f, {x, a, b},
                 PlotStyle- > {{Thickness[0.01], GrayLevel[0.3]}},
                 DisplayFunction- > Identity];
          recs = Table[Rectangle[
                   {z[k - 1], 0}, {z[k], f/.x- > z[k - 1]}], {k, 1, n}];
          ls = Table[Line[{{z[k - 1], 0}, {z[k - 1], f/.x- > z[k - 1]},
                     {z[k], f/.x- > z[k - 1]}, {z[k], 0}}], {k, 1, n}];
          Show[Graphics[{GrayLevel[0.8], recs}],
                Graphics[ls], p1, opts, Axes- > Automatic,
                DisplayFunction- > $DisplayFunction]]
```

```
In[208]:= rightbox[f_, a_, b_, n_, opts___] :=
            Module[{z, p1, recs, ls},
          z[k_] = a + (b - a) k/n;
          p1 = Plot[f, {x, a, b},
                 PlotStyle- > {{Thickness[0.01], GrayLevel[0.3]}},
                 DisplayFunction- > Identity];
          recs = Table[Rectangle[
                   {z[k - 1], 0}, {z[k], f/.x- > z[k]}], {k, 1, n}];
          ls = Table[Line[{{z[k - 1], 0}, {z[k - 1], f/.x- > z[k]},
                     {z[k], f/.x- > z[k]}, {z[k], 0}}], {k, 1, n}];
          Show[Graphics[{GrayLevel[0.8], recs}],
                Graphics[ls], p1, opts, Axes- > Automatic,
                DisplayFunction- > $DisplayFunction]]
```

```
In[209]:= middlebox[f_, a_, b_, n_, opts___] :=
            Module[{z, p1, recs, ls},
           z[k_] = a + (b - a) k/n;
           p1 = Plot[f, {x, a, b},
                  PlotStyle-> {{Thickness[0.01], GrayLevel[0.3]}},
                  DisplayFunction-> Identity];
           recs = Table[Rectangle[{z[k - 1], 0},
                    {z[k], f/.x-> 1/2(z[k - 1] + z[k])}], {k, 1, n}];
           ls = Table[Line[
                    {{z[k - 1], 0}, {z[k - 1], f/.x-> 1/2(z[k - 1] + z[k])},
                      {z[k], f/.x-> 1/2(z[k - 1] + z[k])}, {z[k], 0}}],
                  {k, 1, n}];
           Show[Graphics[{GrayLevel[0.8], recs}],
                Graphics[ls], p1, opts, Axes-> Automatic,
                DisplayFunction-> $DisplayFunction]]

In[210]:= somegraphs = {{leftbox[f[x], 0, 3/2, 4,
                  DisplayFunction-> Identity], middlebox[f[x], 0,
                  3/2, 4, DisplayFunction-> Identity], rightbox[
                  f[x], 0, 3/2, 4, DisplayFunction-> Identity]},
              {leftbox[f[x], 0, 3/2, 16, DisplayFunction-> Identity],
                middlebox[f[x], 0, 3/2,
                  16, DisplayFunction-> Identity], rightbox[
                  f[x], 0, 3/2, 16, DisplayFunction-> Identity]},
              {leftbox[f[x], 0, 3/2, 32, DisplayFunction-> Identity],
                middlebox[f[x], 0, 3/2, 32,
                  DisplayFunction-> Identity], rightbox[f[x],
                  0, 3/2, 32, DisplayFunction-> Identity]}};
           Show[GraphicsArray[somegraphs]]
```

Notice that as n increases, the under approximations increase while the upper approximations decrease.

These graphs help convince us that the limit of the sum as $n \to \infty$ of the areas of the inscribed and circumscribed rectangles is the same. We compute the exact value of (2.5) with `leftsum`, evaluate and simplify the sum with `Simplify`, and compute the limit as $n \to \infty$ with `Limit`, We see that the limit is 9.

```
In[211]:= ls = leftsum[f[x], 0, 3/2, n]

          ls2 = Simplify[ls]

          Limit[ls2, n-> ∞]
```

Out[211]= $-\frac{27\left(n - n^3 - n\ (1 + n) + \frac{1}{6}\ n\ (1 + n)\ (1 + 2\ n)\right)}{2\ n^3}$

Out[211]= $\frac{9\ (-1 + 3\ n + 4\ n^2)}{4\ n^2}$

Out[211]= 9

Similar calculations are carried out for (2.6) and again we see that the limit is 9. We conclude that the exact value of the area is 9.

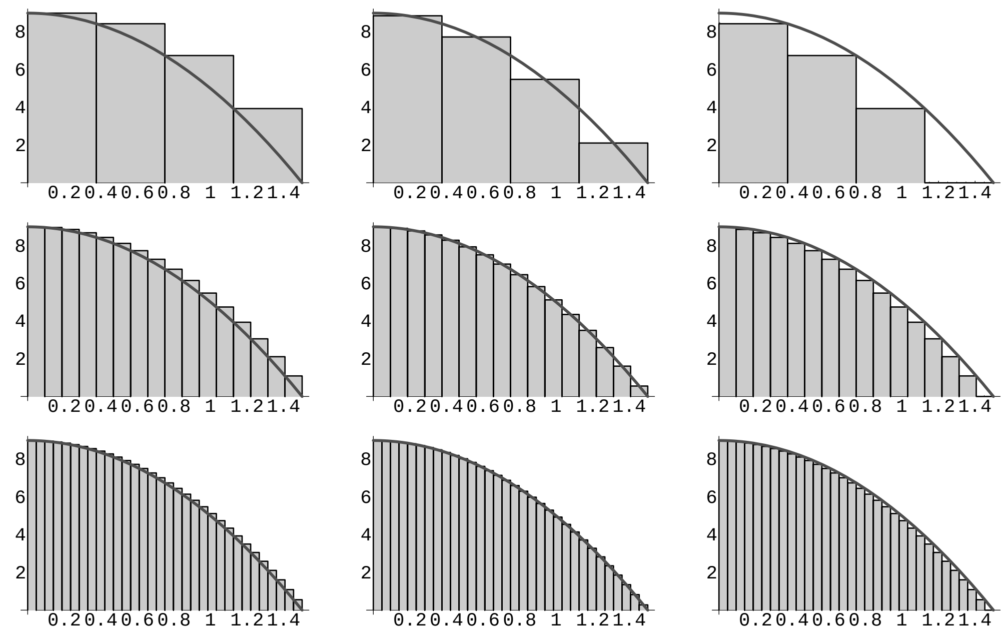


Figure 2.14: $f(x)$ with 4, 16, and 32 rectangles

```
In[212]:= rs = rightsum[f[x], 0, 3/2, n]

          rs2 = Simplify[rs]

          Limit[rs2, n-> ∞]
```

Out[212]= $-\dfrac{27\left(-n^3+\frac{1}{6}\,n\,(1+n)\,(1+2\,n)\right)}{2\,n^3}$

Out[212]= $\dfrac{9\,(-1-3\,n+4\,n^2)}{4\,n^2}$

Out[212]= 9

For illustrative purposes, we confirm this result with `middlesum`.

```
In[213]:= ms = middlesum[f[x], 0, 3/2, n]

          ms2 = Simplify[ms]

          Limit[ms2, n-> ∞]
```

Out[213]= $-\dfrac{27\left(n-4\,n^3-2\,n\,(1+n)+\frac{2}{3}\,n\,(1+n)\,(1+2\,n)\right)}{8\,n^3}$

Out[213]= $9+\dfrac{9}{8\,n^2}$

Out[213]= 9

□

2.3.2 The Definite Integral

In integral calculus courses, we formally learn that the **definite integral** of the function $y = f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b f(x)\,dx = \lim_{|P|\to 0} \sum_{k=1}^{n} f\left(x_k{}^*\right)\Delta x_k, \tag{2.8}$$

provided that the limit exists. In (2.8), $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ is a partition of $[a, b]$, $|P|$ is the **norm** of P,

$$|P| = \max\{x_k - x_{k-1} | k = 1, 2, \ldots, n\},$$

$\Delta x_k = x_k - x_{k-1}$, and $x_k{}^* \in \left[x_{k-1}, x_k\right]$.

The Fundamental Theorem of Calculus provides the fundamental relationship between differentiation and integration.

Theorem 5 (The Fundamental Theorem of Calculus). *Suppose that $y = f(x)$ is continuous on $[a, b]$.*

1. *If $F(x) = \int_a^x f(t)\,dt$, then F is an antiderivative of f: $F'(x) = f(x)$.*
2. *If G is any antiderivative of f, then $\int_a^b f(x)\,dx = G(b) - G(a)$.*

Mathematica's `Integrate` command can compute many definite integrals. The command

```
Integrate[f[x],{x,a,b}]
```

attempts to compute $\int_a^b f(x)\,dx$. Because integration is a fundamentally difficult procedure, it is easy to create integrals for which the exact value cannot be found explicitly. In those cases, use `N` to obtain an approximation of its value or obtain a numerical approximation of the integral directly with

```
NIntegrate[f[x],{x,a,b}].
```

Example 38. Evaluate (a) $\int_1^4 \left(x^2+1\right)/\sqrt{x}\,dx$; (b) $\int_0^{\sqrt{\pi/2}} x\cos x^2\,dx$; (c) $\int_0^{\pi} e^{2x}\sin^2 2x\,dx$; (d) $\int_0^1 \frac{2}{\sqrt{\pi}}e^{-x^2}\,dx$; and (e) $\int_{-1}^0 \sqrt[3]{u}\,du$.

Solution. We evaluate (a)-(c) directly with `Integrate`.

```
In[214]:= Integrate[(x^2 + 1)/Sqrt[x], {x, 1, 4}]
Out[214]= 72/5
In[215]:= Integrate[x Cos[x^2], {x, 0, Sqrt[π/2]}]
```

Out[215]= $\frac{1}{2}$

```
In[216]:= Integrate[Exp[2x] Sin[2x]^2, {x, 0, π}]
```

Out[216]= $-\frac{1}{5}+\frac{e^{2\pi}}{5}$

For (d), the result returned is in terms of the **error function**, `Erf[x]`, that is defined by the integral

$$\texttt{Erf[x]} = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

```
In[217]:= Integrate[2/Sqrt[π] Exp[-x^2], {x, 0, 1}]
Out[217]= Erf[1]
```

We use `N` to obtain an approximation of the value of the definite integral.

```
In[218]:= Integrate[2/Sqrt[π] Exp[-x^2], {x, 0, 1}]//N
Out[218]= 0.842701
```

(e) Recall that Mathematica does not return a real number when we compute odd roots of negative numbers so the following result would be surprising to many students in an introductory calculus course because it contains imaginary numbers.

```
In[219]:= Integrate[u^(1/3), {u, -1, 0}]
```

Out[219]= $\frac{3}{4}\,(-1)^{1/3}$

Therefore, we load the `RealOnly` package contained in the **Miscellaneous** directory so that Mathematica. returns the real-valued third root of *u*.

```
In[220]:= << Miscellaneous`RealOnly`

In[221]:= Integrate[u^(1/3), {u, -1, 0}]
```

Out[221]= $-\frac{3}{4}$

□

Improper integrals are computed using `Integrate` in the same way as other definite integrals.

Example 39. Evaluate (a) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$; (b) $\int_{-\infty}^{\infty} \frac{2}{\sqrt{\pi}} e^{-x^2}\,dx$; (c) $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}} dx$; (d) $\int_0^{\infty} \frac{1}{x^2+x^4}\,dx$; (e) $\int_2^4 \frac{1}{\sqrt[3]{(x-3)^2}}\,dx$; and (f) $\int_{-\infty}^{\infty} \frac{1}{x^2+x-6}\,dx$.

Solution. (a) This is an improper integral because the integrand is discontinuous on the interval $[0, 1]$ but we see that the improper integral converges to -4.

```
In[222]:= Integrate[Log[x]/Sqrt[x], {x, 0, 1}]
Out[222]= -4
```

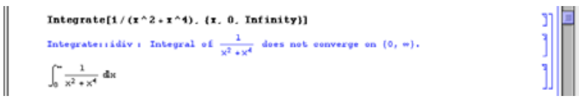
(b) This is an improper integral because the interval of integration is infinite but we see that the improper integral converges to 2.

```
In[223]:= Integrate[2/Sqrt[π] Exp[-x^2], {x, -∞, ∞}]
Out[223]= 2
```

(c) This is an improper integral because the integrand is discontinuous on the interval of integration and because the interval of integration is infinite but we see that the improper integral converges to $\pi/2$.

```
In[224]:= Integrate[1/(x Sqrt[x^2 - 1]), {x, 1, ∞}]
Out[224]= π/2
```

(d) As with (c), this is an improper integral because the integrand is discontinuous on the interval of integration and because the interval of integration is infinite but we see that the improper integral diverges to ∞.



(e) Recall that Mathematica does not return a real number when we compute odd roots of negative numbers so the following result would be surprising to many students in an introductory calculus course because it contains imaginary numbers.

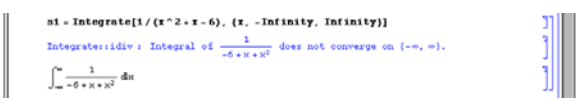
```
In[225]:= Integrate[1/(x - 3)^(2/3), {x, 2, 4}]
Out[225]= 3 - 3 (-1)^(1/3)
```

Therefore, we load the `RealOnly` package contained in the **Miscellaneous** directory so that Mathematica returns the real-valued third root of $x - 3$.

```
In[226]:= << Miscellaneous‘RealOnly‘
```

```
In[227]:= Integrate[1/(x - 3)^(2/3), {x, 2, 4}]
Out[227]= 6
```

(f) In this case, Mathematica warns us that the improper integral diverges.



To help us understand why the improper integral diverges, we note that that $\frac{1}{x^2+x-6} = \frac{1}{5}\left(\frac{1}{x-2} - \frac{1}{x+3}\right)$ and

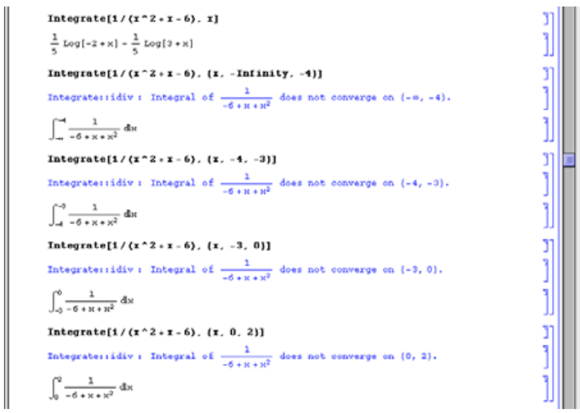
$$\int \frac{1}{x^2 + x - 6}\, dx = \int \frac{1}{5}\left(\frac{1}{x-2} - \frac{1}{x+3}\right) dx = \frac{1}{5} \ln\left(\frac{x-2}{x+3}\right) + C.$$

```
In[228]:= Integrate[1/(x^2 + x - 6), x]
Out[228]= 1/5 Log[-2 + x] - 1/5 Log[3 + x]
```

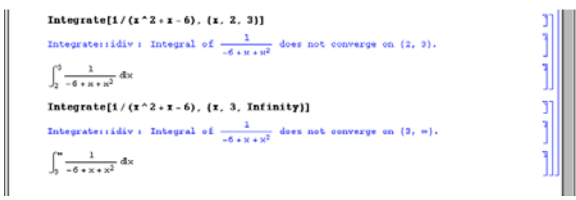
Hence the integral is improper because the interval of integration is infinite and because the integrand is discontinuous on the interval of integration so

$$\int_{-\infty}^{\infty} \frac{1}{x^2+x-6}\,dx = \int_{-\infty}^{-4} \frac{1}{x^2+x-6}\,dx + \int_{-4}^{-3} \frac{1}{x^2+x-6}\,dx + \int_{-3}^{0} \frac{1}{x^2+x-6}\,dx + \int_{0}^{2} \frac{1}{x^2+x-6}\,dx + \int_{2}^{3} \frac{1}{x^2+x-6}\,dx + \int_{3}^{\infty} \frac{1}{x^2+x-6}\,dx \tag{2.9}$$

Evaluating each of these integrals,



```
Integrate[1/(x^2+x-6), x]
1/5 Log[-2+x] - 1/5 Log[3+x]
Integrate[1/(x^2+x-6), {x, -Infinity, -4}]
Integrate::idiv : Integral of 1/(-6+x+x^2) does not converge on {-∞, -4}.
∫_{-∞}^{-4} 1/(-6+x+x^2) dx
Integrate[1/(x^2+x-6), {x, -4, -3}]
Integrate::idiv : Integral of 1/(-6+x+x^2) does not converge on {-4, -3}.
∫_{-4}^{-3} 1/(-6+x+x^2) dx
Integrate[1/(x^2+x-6), {x, -3, 0}]
Integrate::idiv : Integral of 1/(-6+x+x^2) does not converge on {-3, 0}.
∫_{-3}^{0} 1/(-6+x+x^2) dx
Integrate[1/(x^2+x-6), {x, 0, 2}]
Integrate::idiv : Integral of 1/(-6+x+x^2) does not converge on {0, 2}.
∫_{0}^{2} 1/(-6+x+x^2) dx
```



```
Integrate[1/(x^2+x-6), {x, 2, 3}]
Integrate::idiv : Integral of 1/(-6+x+x^2) does not converge on {2, 3}.
∫_{2}^{3} 1/(-6+x+x^2) dx
Integrate[1/(x^2+x-6), {x, 3, Infinity}]
Integrate::idiv : Integral of 1/(-6+x+x^2) does not converge on {3, ∞}.
∫_{3}^{∞} 1/(-6+x+x^2) dx
```

we conclude that the improper integral diverges because at least one of the improper integrals in (2.9) diverges. □

In many cases, Mathematica can help illustrate the steps carried out when computing integrals using standard methods of integration like u-substitutions and integration by parts.

Example 40. Evaluate (a) $\int_e^{e^3} \frac{1}{x\sqrt{\ln x}}\,dx$ and (b) $\int_0^{\pi/4} x\sin 2x\,dx$.

Solution. (a) We let $u = \ln x$. Then, $du = \frac{1}{x}dx$ so $\int_e^{e^3} \frac{1}{x\sqrt{\ln x}}\,dx = \int_1^3 \frac{1}{\sqrt{u}}du = \int_1^3 u^{-1/2}du$, which we evaluate with `Integrate`.

The new lower limit of integration is 1 because if $x = e$, $u = \ln e = 1$. The new upper limit of integration is 3 because if $x = e^3$, $u = \ln e^3 = 3$.

```
In[229]:= Integrate[1/Sqrt[u], {u, 1, 3}]
Out[229]= -2 + 2 √3
```

To evaluate (b), we let $u = x \Rightarrow du = dx$ and $dv = \sin 2x\,dx \Rightarrow v = -\frac{1}{2}\cos 2x$.

```
In[230]:= u = x;
          dv = Sin[2x];
In[231]:= du = D[x, x]

          v = Integrate[Sin[2x], x]
Out[231]= 1
Out[231]= -1/2 Cos[2 x]
In[232]:= v du
Out[232]= -1/2 Cos[2 x]
```

The results mean that

$$\begin{aligned}\int_0^{\pi/4} x \sin 2x \, dx &= -\frac{1}{2} x \cos 2x \Big]_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} \cos 2x \, dx \\ &= 0 + \frac{1}{2} \int_0^{\pi/4} \cos 2x \, dx.\end{aligned}$$

The resulting indefinite integral is evaluated with `Integrate`

```
In[233]:= u v - Integrate[v du, x]
Out[233]= -1/2 x Cos[2 x] + 1/4 Sin[2 x]
In[234]:= Integrate[x Sin[2x], x]
Out[234]= 1/4 (-2 x Cos[2 x] + Sin[2 x])
```

and the definite integral is evaluated with `Integrate`.

```
Integrate[x Sin[2 x], {x, 0, Pi / 4}]
1/4
```

□

2.3.3 Approximating Definite Integrals

Because integration is a fundamentally difficult procedure, Mathematica is unable to compute a ”closed form” of the value of many definite integrals. In these cases, numerical integration can be used to obtain an approximation of the definite integral using `N` together with `Integrate` or `NIntegrate`.

Example 41. Evaluate

$$\int_0^{\sqrt[3]{\pi}} e^{-x^2} \cos x^3 \, dx.$$

Solution. In this case, Mathematica is unable to evaluate the integral with `Integrate`.

```
In[235]:= i1 = Integrate[Exp[-x^2] Cos[x^3], {x, 0, π^(1/3)}]
```

```
Out[235]= ∫_0^(π^(1/3)) e^(-x^2) Cos[x^3] ⅆx
```

An approximation is obtained with `N`.

```
In[236]:= N[i1]
Out[236]= 0.701566
```

Instead of using `Integrate` followed by `N`, you can use `NIntegrate` to numerically evaluate many integrals.

```
NIntegrate[f[x],{x,a,b}]
```

attempts to approximate $\int_a^b f(x)\,dx$. Thus, entering

```
In[237]:= NIntegrate[Exp[-x^2] Cos[x^3], {x, 0, π^(1/3)}]
Out[237]= 0.701566
```

returns the same result as that obtained using `Integrate` followed by `N`. □

In some cases, you may wish to investigate particular numerical methods that can be used to approximate integrals. If needed you can redefine the functions `leftsum`, `middlesum`, and `rightsum` that were discussed previously. In addition we define the functions `simpson`, which implements Simpson's rule, and `trapezoid`, which implements the trapezoidal rule, in the following example that can be used to investigate approximations of definite integrals using those numerical methods.

Example 42. Let

$$f(x) = e^{-(x-3)^2 \cos(4(x-3))}.$$

(a) Graph $y = f(x)$ on the interval $[1, 5]$. Use (b) Simpson's rule with $n = 4$, (c) the trapezoidal rule with $n = 4$, and (d) the midpoint rule with $n = 4$ to approximate $\int_1^5 f(x)\,dx$.

Solution. We define f, and then graph $y = f(x)$ on the interval $[1, 5]$ with `Plot` in Figure 2.15.

```
In[238]:= f[x_] = Exp[-(x - 3)^2 Cos[4(x - 3)]];
          Plot[f[x], {x, 1, 5}]
```

After defining `simpson` and `trapezoid`,

```
In[239]:= simpson[f_, a_, b_, n_] := Module[{z, h},
          h = (b - a)/n;
          z[k_] = a + h k;
          f0 = f/.x- > z[0];
          fn = f/.x- > z[n];
          h/3 (f0 + fn) +
               h/3 Sum[(3 + (-1)^(k + 1))f /. x- > z[k], {k, 1, n - 1}]
          ]
```

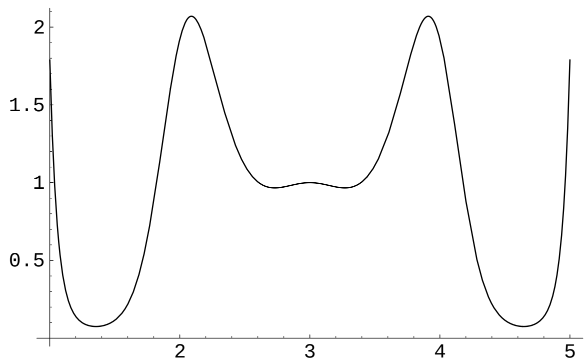


Figure 2.15: $f(x)$ for $1 \le x \le 5$

```
In[240]:= trapezoid[f_, a_, b_, n_] := Module[{z, h},
          h = (b - a)/n;
          z[k_] = a + h k;
          f0 = f/.x- > z[0];
          fn = f/.x- > z[n];
          h/2 (f0 + fn) + h Sum[f /. x- > z[k], {k, 1, n - 1}]
          ]
```

Be sure to redefine `middlesum` if you have not already used it during your current Mathematica session before executing the following commands.

we use these functions and `middlesum`, which was defined earlier, to approximate $\int_1^5 f(x)\,dx$ using $n = 4$ rectangles. In each case, `N` is used to evaluate the sum.

```
In[241]:= s1 = simpson[f[x], 1, 5, 4]

          N[s1]

          t1 = trapezoid[f[x], 1, 5, 4]

          N[t1]

          m1 = middlesum[f[x], 1, 5, 4]

          N[m1]
```

Out[241]= $\frac{2}{3}\ e^{-4\ \text{Cos}[8]} + \frac{1}{3}\ \left(2 + 8\ e^{-\text{Cos}[4]}\right)$

Out[241]= 6.9865

Out[241]= $1 + 2\ e^{-\text{Cos}[4]} + e^{-4\ \text{Cos}[8]}$

Out[241]= 6.63468

Out[241]= $2\ e^{-\frac{\text{Cos}[2]}{4}} + 2\ e^{-\frac{9\ \text{Cos}[6]}{4}}$

Out[241]= 2.44984

We obtain an accurate approximation of the value of the integral using `NIntegrate`.

```
In[242]:= NIntegrate[f[x], {x, 1, 5}]
Out[242]= 3.761
```

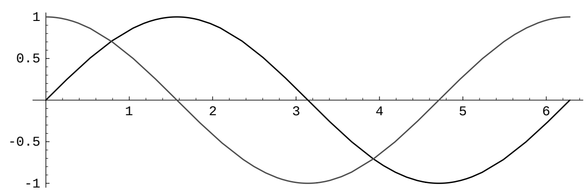


Figure 2.16: $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$

Notice that with $n = 4$ rectangles, the midpoint rule gives the best approximation. However, as n increases, Simpson's rule gives a better approximation as we see using $n = 50$ rectangles.

```
In[243]:= simpson[f[x], 1, 5, 50]//N

          trapezoid[f[x], 1, 5, 50]//N

          middlesum[f[x], 1, 5, 50]//N
Out[243]= 3.76445
Out[243]= 3.7913
Out[243]= 3.74623
```

□

2.3.4 Area

Suppose that $y = f(x)$ and $y = g(x)$ are continuous on $[a, b]$ and that $f(x) \geq g(x)$ for $a \leq x \leq b$. Then, the **area** of the region bounded by the graphs of $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx. \tag{2.10}$$

Example 43. Find the area between the graphs of $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$.

Solution. We graph $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$ in Figure 2.16. The graph of $y = \cos x$ is gray.

```
In[244]:= Plot[{Sin[x], Cos[x]}, {x, 0, 2π},
            PlotStyle->{GrayLevel[0], GrayLevel[0.3]},
            AspectRatio->Automatic]
```

To find the upper and lower limits of integration, we must solve the equation $\sin x = \cos x$ for x.

```
In[245]:= Solve[Sin[x] == Cos[x], x]
```

```
Solve :: "ifun" : "Inversefunctionsarebeingused
     bySolve, so some solutions may not be found."

Set :: "write" : "TagPower in DownValues[
     Power] is Protected."
```

Out[245]= $\left\{\left\{x \to -\frac{3\pi}{4}\right\}, \left\{x \to \frac{\pi}{4}\right\}\right\}$

Thus, for $0 \le x \le 2\pi$, $\sin x = \cos x$ if $x = \pi/4$ or $x = 5\pi/4$. Hence, the area of the region between the graphs is given by

$$A = \int_0^{\pi/4} [\cos x - \sin x]\, dx + \int_{\pi/4}^{5\pi/4} [\sin x - \cos x]\, dx + \int_{5\pi/4}^{2\pi} [\cos x - \sin x]\, dx. \tag{2.11}$$

Notice that if we take advantage of symmetry we can simplify (2.11) to

$$A = 2\int_{\pi/4}^{5\pi/4} [\sin x - \cos x]\, dx. \tag{2.12}$$

We evaluate (2.12) with `Integrate` to see that the area is $4\sqrt{2}$.

```
In[246]:= 2 Integrate[Sin[x] - Cos[x], {x, π/4, 5π/4}]
Out[246]= 4 √2
```

□

In cases when we cannot calculate the points of intersection of two graphs exactly, we can frequently use `FindRoot` to approximate the points of intersection.

Example 44. Let

$$p(x) = \frac{3}{10}x^5 - 3x^4 + 11x^3 - 18x^2 + 12x + 1$$

and

$$q(x) = -4x^3 + 28x^2 - 56x + 32.$$

Approximate the area of the region bounded by the graphs of p and q.

Solution. After defining p and q, we graph them on the interval $[-1, 5]$ in Figure 2.17 to obtain an initial guess of the intersection points of the two graphs.

```
In[247]:= p[x_] = 3/10x^5 - 3x^4 + 11x^3 - 18x^2 + 12x + 1;
          q[x_] = -4x^3 + 28x^2 - 56x + 32;
          Plot[{p[x], q[x]}, {x, -1, 5}, PlotRange- > {-15, 20},
            PlotStyle- > {GrayLevel[0], GrayLevel[0.3]}]
```

The x-coordinates of the three intersection points are the solutions of the equation $p(x) = q(x)$. Although Mathematica can solve this equation exactly, approximate solutions are more useful for the problem and obtained with `FindRoot`.

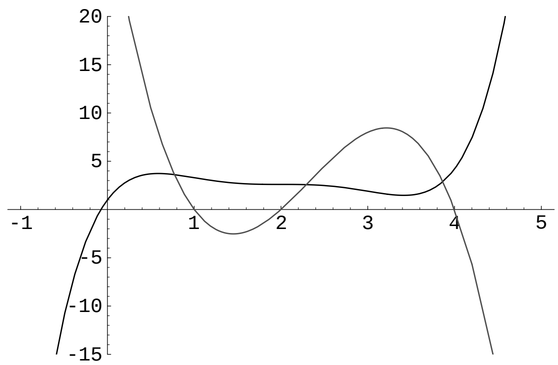


Figure 2.17: p and q on the interval $[-1, 5]$

```
In[248]:= FindRoot[p[x] == q[x], {x, 1}]

          FindRoot[p[x] == q[x], {x, 2}]

          FindRoot[p[x] == q[x], {x, 4}]
Out[248]= {x → 0.772058}
Out[248]= {x → 2.29182}
Out[248]= {x → 3.86513}
```

All three `FindRoot` commands can be combined together if we use `Map` as illustrated next.

```
In[249]:= intpts = Map[FindRoot[p[x] == q[x], {x, #}]&, {1, 2, 4}]
Out[249]= {{x → 0.772058}, {x → 2.29182},
            {x → 3.86513}}
In[250]:= intpts[[1, 1, 2]]
Out[250]= 0.772058
```

Using the roots to the equation $p(x) = q(x)$ and the graph we see that $p(x) \geq q(x)$ for $0.772 \leq x \leq 2.292$ and $q(x) \geq p(x)$ for $2.292 \leq x \leq 3.865$. Hence, an approximation of the area bounded by p and q is given by the sum

$$\int_{0.772}^{2.292} [p(x) - q(x)] \; dx + \int_{2.292}^{3.865} [q(x) - p(x)] \; dx.$$

These two integrals are computed with `NIntegrate`.

```
In[251]:= intone = NIntegrate[p[x] - q[x],
              {x, intpts[[1, 1, 2]], intpts[[2, 1, 2]]}]

          inttwo = NIntegrate[q[x] - p[x],
              x, intpts[[2, 1, 2]], intpts[[3, 1, 2]]}]
Out[251]= 5.26912
Out[251]= 6.92599
```

and added to see that the area is approximately 12.195.

```
In[252]:= intone + inttwo
Out[252]= 12.1951
```

□

Parametric Equations

Graphically, y is a function of x, $y = y(x)$, if the graph of $y = y(x)$ passes the vertical line test.

If the curve, C, defined parametrically by $x = x(t)$, $y = y(t)$, $a \le t \le b$ is a nonnegative continuous function of x and $x(a) < x(b)$ the area under the graph of C and above the x-axis is

$$\int_{x(a)}^{x(b)} y\,dx = \int_a^b y(t)x'(t)dt.$$

Example 45 (The Astroid). Find the area enclosed by the **astroid** $x = \sin^3 t$, $y = \cos^3 t$, $0 \le t \le 2\pi$.

Solution. We begin by defining x and y and then graphing the asteroid with `ParametricPlot` in Figure 2.18.

```
In[253]:= x[t_] = Sin[t]^3;
          y[t_] = Cos[t]^3;
          ParametricPlot[
            {x[t], y[t]}, {t, 0, 2π}, AspectRatio- > Automatic]
```

Observe that $x(0) = 0$ and $x(\pi/2) = 1$ and the graph of the asteroid in the first quadrant is given by $x = \sin^3 t$, $y = \cos^3 t$, $0 \le t \le \pi/2$. Hence, the area of the astroid in the first quadrant is given by

$$\int_0^{\pi/2} y(t)x'(t)\,dt = 3\int_0^{\pi/2} \sin^2 t\cos^4 t\,dt$$

and the total area is given by

$$A = 4\int_0^{\pi/2} y(t)x'(t)\,dt = 3\int_0^{\pi/2} \sin^2 t\cos^4 t\,dt = \frac{3}{8}\pi \approx 1.178,$$

which is computed with `Integrate` and then approximated with `N`.

```
In[254]:= area = 4 Integrate[y[t] x'[t], {t, 0, π/2}]
Out[254]= 3 π
          ---
           8
In[255]:= N[area]
Out[255]= 1.1781
```

□

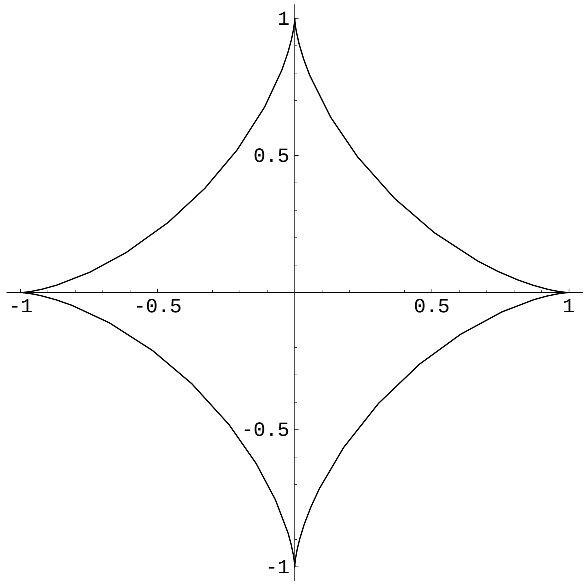


Figure 2.18: The astroid $x = \sin^3 t$, $y = \cos^3 t$, $0 \leq t \leq 2\pi$

Polar Coordinates

For problems involving ”circular symmetry” it is often easier to work in polar coordinates. The relationship between (x, y) in rectangular coordinates and (r, θ) in polar coordinates is given by

$$x = r\cos\theta \qquad y = r\sin\theta$$

and

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

If $r = f(\theta)$ is continuous and non-negative for $\alpha \leq \theta \leq \beta$, then the **area** A of the region enclosed by the graphs of $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$ is

$$A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 \, d\theta = \frac{1}{2}\int_\alpha^\beta r^2 \, d\theta.$$

Example 46. The **Lemniscate of Bernoulli** is given by

$$\left(x^2 + y^2\right)^2 = a^2\left(x^2 - y^2\right),$$

where a is a constant. (a) Graph the Lemniscate of Bernoulli if $a = 2$. (b) Find the area of the region bounded by the Lemniscate of Bernoulli.

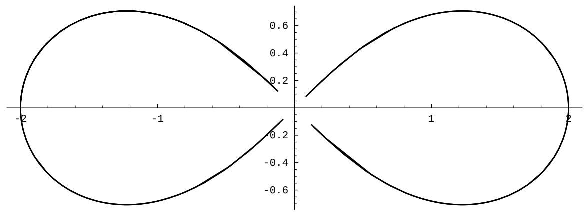


Figure 2.19: The Lemniscate

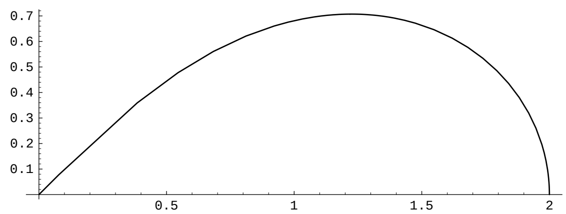


Figure 2.20: The portion of the Lemniscate in quadrant 1

Solution. This problem is much easier solved in polar coordinates so we first convert the equation from rectangular to polar coordinates with `/.` and then solve for r with `Solve`.

```
In[256]:= lofb = (x^2 + y^2)^2 == a^2(x^2 - y^2);
          topolar = lofb/.{x- > r Cos[t], y- > r Sin[t]}
Out[256]= (r^2 Cos[t]^2 + r^2 Sin[t]^2)^2 == a^2 (r^2 Cos[t]^2 - r^2 Sin[t]^2)
```

```
In[257]:= Solve[topolar, r]//Simplify
Out[257]= {{r → 0}, {r → 0}, {r → -a √Cos[2 t]}, {r → a √Cos[2 t]}}
```

These results indicate that an equation of the Lemniscate in polar coordinates is $r^2 = a^2 \cos 2\theta$. The graph of the Lemniscate is then generated in Figure 2.19 using `PolarPlot`, which is contained in the `Graphics` package located in the **Graphics** directory.

```
In[258]:= << Graphics`Graphics`

          PolarPlot[{-2 Sqrt[Cos[2t]], 2Sqrt[Cos[2t]]}, {t, 0, 2π}]
```

The portion of the Lemniscate in quadrant one is obtained by graphing $r = 2\cos 2\theta$, $0 \le \theta \le \pi/4$.

```
In[259]:= PolarPlot[2Sqrt[Cos[2t]], {t, 0, π/4}]
```

Then, taking advantage of symmetry, the area of the Lemniscate is given by

$$A = 2 \cdot \frac{1}{2} \int_{-\pi/4}^{\pi/4} r^2 \, d\theta = 2 \int_0^{\pi/4} r^2 \, d\theta = 2 \int_0^{\pi/4} a^2 \cos 2\theta \, d\theta = a^2,$$

which we calculate with `Integrate`.

```
In[260]:= Integrate[2 a^2 Cos[2 t], {t, 0, π/4}]
Out[260]= a^2
```

□

2.3.5 Arc Length

Let $y = f(x)$ be a function for which $f'(x)$ is continuous on an interval $[a, b]$. Then the **arc length** of the graph of $y = f(x)$ from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}dx \tag{2.13}$$

The resulting definite integrals used for determining arc length are usually difficult to compute because they involve a radical. In these situations, Mathematica is helpful with approximating solutions to these types of problems.

Example 47. Find the length of the graph of

$$y = \frac{x^4}{8} + \frac{1}{4x^2}$$

from (a) $x = 1$ to $x = 2$ and (b) from $x = -2$ to $x = -1$.

Solution. With no restrictions on the value of x, $\sqrt{x^2} = |x|$. Notice that Mathematica does not automatically algebraically simplify $\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$ because Mathematica does not know if x is positive or negative.

```
In[261]:= y[x_] = x^4/8 + 1/(4x^2);
          i1 = Factor[y'[x]^2 + 1]
Out[261]= (1 + x^2)^2 (1 - x^2 + x^4)^2 / (4 x^6)
In[262]:= i2 = PowerExpand[Sqrt[i1]]
Out[262]= (1 + x^2) (1 - x^2 + x^4) / (2 x^3)
```

`PowerExpand[expr]` simplifies radicals in the expression `expr`.

In fact, for (b), x is negative so

$$\frac{1}{2}\sqrt{\frac{\left(x^6 + 1\right)^2}{x^6}} = -\frac{1}{2}\frac{x^6 + 1}{x^3}.$$

Mathematica simplifies

$$\frac{1}{2}\sqrt{\frac{\left(x^6 + 1\right)^2}{x^6}} = \frac{1}{2}\frac{x^6 + 1}{x^3}.$$

and correctly evaluates the arc length integral (2.13) for (a).

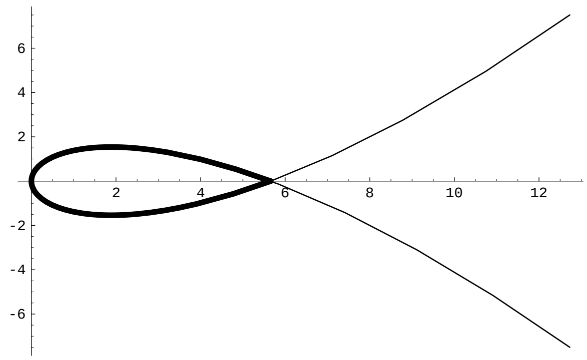


Figure 2.21: $x = \sqrt{2}t^2$, $y = 2t - \frac{1}{2}t^3$

```
In[263]:= Integrate[Sqrt[y′[x]^2 + 1], {x, 1, 2}]
Out[263]= 33/16
```

For (b), we compute the arc length integral (2.13).

```
In[264]:= Integrate[Sqrt[y′[x]^2 + 1], {x, -2, -1}]
Out[264]= 33/16
```

As we expect, both values are the same. □

Parametric Equations

C is **smooth** if both $x'(t)$ and $y'(t)$ are continuous on (a, b) and not simultaneously zero for $t \in (a, b)$.

If the smooth curve, C, defined parametrically by $x = x(t)$, $y = y(t)$, $t \in [a, b]$ is traversed exactly once as t increases from $t = a$ to $t = b$, the arc length of C is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{2.14}$$

Example 48. Find the length of the graph of $x = \sqrt{2}t^2$, $y = 2t - \frac{1}{2}t^3$, $-2 \le t \le 2$.

Solution. For illustrative purposes, we graph $x = \sqrt{2}t^2$, $y = 2t - \frac{1}{2}t^3$ for $-3 \le t \le 3$ (in black) and $-2 \le t \le 2$ (in thick black) in Figure 2.21.

```
In[265]:= x[t_] = t^2 Sqrt[2]; y[t_] = 2t - 1/2t^3;
          p1 = ParametricPlot[{x[t], y[t]},
              {t, -3, 3}, DisplayFunction- > Identity];
          p2 = ParametricPlot[{x[t], y[t]}, {t, -2, 2}, PlotStyle- >
                Thickness[0.01], DisplayFunction- > Identity];
          Show[p1, p2, DisplayFunction- > $DisplayFunction,
            PlotRange- > All]
```

Mathematica is able to compute the exact value of the arc length (2.14) although the result is quite complicated.

In[266]:= **Factor[x′[t]^2 + y′[t]^2]**

Out[266]= $\frac{1}{4}\left(4-4\,t+3\,t^2\right)\left(4+4\,t+3\,t^2\right)$

In[267]:= **i1 = Integrate[2 Sqrt[x′[t]^2 + y′[t]^2], {t, 0, 2}]**

Out[267]=
$$\frac{1}{18}\Bigg(96\sqrt{3}-\Bigg(32\,i\sqrt{6\left(1+\frac{1}{2}\left(2-4\,i\sqrt{2}\right)\right)}\sqrt{1+\frac{1}{2}\left(2+4\,i\sqrt{2}\right)}\,\text{EllipticE}\Big[i\,\text{ArcSinh}\Big[\sqrt{\frac{1}{2}\left(2+4\,i\sqrt{2}\right)}\Big],-1+\frac{1}{9}\left(2-4\,i\sqrt{2}\right)\Big]\Bigg)\Big/\left(\left(2+4\,i\sqrt{2}\right)^{3/2}\left(-1+\frac{1}{9}\left(2-4\,i\sqrt{2}\right)\right)\right)-\frac{1}{\sqrt{2+4\,i\sqrt{2}}}\Bigg(16\,i\sqrt{6\left(1+\frac{1}{2}\left(2-4\,i\sqrt{2}\right)\right)}\sqrt{1+\frac{1}{2}\left(2+4\,i\sqrt{2}\right)}\,\text{EllipticF}\Big[i\,\text{ArcSinh}\Big[\sqrt{\frac{1}{2}\left(2+4\,i\sqrt{2}\right)}\Big],-1+\frac{1}{9}\left(2-4\,i\sqrt{2}\right)\Big]\Bigg)+\Bigg(32\,i\sqrt{6\left(1+\frac{1}{2}\left(2-4\,i\sqrt{2}\right)\right)}\sqrt{1+\frac{1}{2}\left(2+4\,i\sqrt{2}\right)}\,\text{EllipticF}\Big[i\,\text{ArcSinh}\Big[\sqrt{\frac{1}{2}\left(2+4\,i\sqrt{2}\right)}\Big],-1+\frac{1}{9}\left(2-4\,i\sqrt{2}\right)\Big]\Bigg)\Big/\left(\left(2+4\,i\sqrt{2}\right)^{3/2}\left(-1+\frac{1}{9}\left(2-4\,i\sqrt{2}\right)\right)\right)\Bigg)$$

A more meaningful approximation is obtained with `N` or using `NIntegrate`.

In[268]:= **N[i1]**

Out[268]= $13.7099-1.18424\,10^{-15}\,i$

In[269]:= **NIntegrate[2 Sqrt[x′[t]^2 + y′[t]^2], {t, 0, 2}]**

Out[269]= 13.7099

We conclude that the arc length is approximately 13.71. □

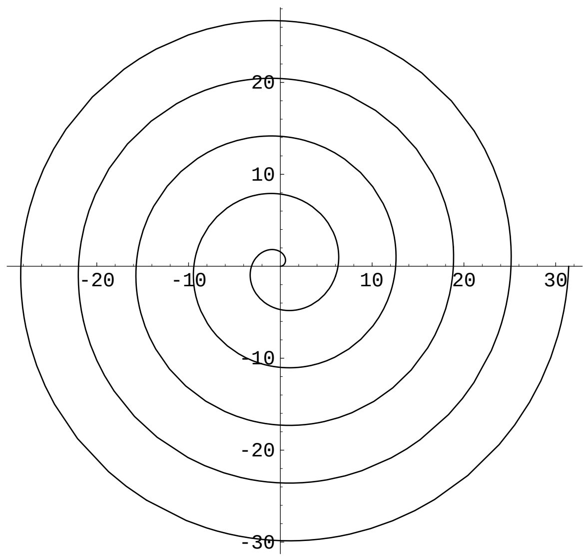


Figure 2.22: $r = \theta$ for $0 \le \theta \le 10\pi$

Polar Coordinates

If the smooth polar curve C given by $r = f(\theta)$, $\alpha \le \theta \le \beta$ is traversed exactly once as θ increases from α to β, the arc length of C is given by

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta \tag{2.15}$$

Example 49. Find the length of the graph of $r = \theta$, $0 \le \theta \le 10\pi$.

Solution. We begin by defining r and then graphing r with `PolarPlot` in Figure 2.22.

```
In[270]:= << Graphics`Graphics`

          r[t_] = t;
          PolarPlot[r[t], {t, 0, 10π}, AspectRatio-> Automatic]
```

Using (2.15), the length of the graph of r is given by $\int_0^{10\pi} \sqrt{1+\theta^2}\, d\theta$. The exact value is computed with `Integrate`

```
In[271]:= ev = Integrate[Sqrt[r'[t]^2 + r[t]^2], {t, 0, 10π}]
```

Out[271]= $5\pi\sqrt{1+100\pi^2} + \frac{1}{2}\text{ArcSinh}[10\pi]$

and then approximated with `N`.

```
In[272]:= N[ev]
```

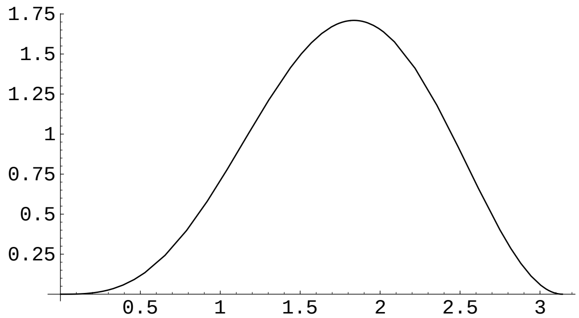


Figure 2.23: $g(x)$ for $0 \le x \le \pi$

```
Out[272]= 495.801
```

We conclude that the length of the graph is approximately 495.8. □

2.3.6 Solids of Revolution

Volume

Let $y = f(x)$ be a non-negative continuous function on $[a, b]$. The **volume** of the solid of revolution obtained by revolving the region bounded by the graphs of $y = f(x)$, $x = a$, $x = b$, and the x-axis about the x-axis is given by

$$V = \pi \int_a^b [f(x)]^2 \, dx. \tag{2.16}$$

If $0 \le a < b$, the **volume** of the solid of revolution obtained by revolving the region bounded by the graphs of $y = f(x)$, $x = a$, $x = b$, and the x-axis about the y-axis is given by

$$V = 2\pi \int_a^b x\, f(x)\, dx \tag{2.17}$$

Example 50. Let $g(x) = x \sin^2 x$. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = g(x)$, $x = 0$, $x = \pi$, and the x-axis about (a) the x-axis; and (b) the y-axis.

Solution. After defining g, we graph g on the interval $[0, \pi]$ in Figure 2.23.

```
In[273]:= g[x_] = x Sin[x]^2;
          Plot[g[x], {x, 0, π}, AspectRatio- > Automatic]
```

The volume of the solid obtained by revolving the region about the x-axis is given by (2.16) while the volume of the solid obtained by revolving the region about the y-axis is given by (2.17). These integrals are computed with `Integrate` and named `xvol` and `yvol`, respectively. `N` is used to approximate each volume.

```
In[274]:= xvol = Integrate[π g[x]^2, {x, 0, π}]

          N[xvol]
Out[274]= 1/256 π (-60 π + 32 π^3)
Out[274]= 9.86295
```

```
In[275]:= yvol = Integrate[2 π x g[x], {x, 0, π}]

          N[yvol]
Out[275]= 1/12 π (-6 π + 4 π^3)
Out[275]= 27.5349
```

We can use `ParametricPlot3D` to visualize the resulting solids by parametrically graphing the equations given by

$$\begin{cases} x = r\cos t \\ y = r\sin t \\ z = g(r) \end{cases}$$

for r between 0 and π and t between $-\pi$ and π to visualize the graph of the solid obtained by revolving the region about the y-axis and by parametrically graphing the equations given by

$$\begin{cases} x = r \\ y = g(r)\cos t \\ z = g(r)\sin t \end{cases}$$

for r between 0 and π and t between $-\pi$ and π to visualize the graph of the solid obtained by revolving the region about the x-axis. (See Figures 2.24 and 2.25.) In this case, we identify the z-axis as the y-axis. Notice that we are simply using polar coordinates for the x and y-coordinates, and the height above the x,y-plane is given by $z = g(r)$ because r is replacing x in the new coordinate system.

```
In[276]:= ParametricPlot3D[{r, g[r] Cos[t], g[r] Sin[t]},
            {r, 0, π}, {t, 0, 2π}, PlotPoints- > {30, 30}]
```

```
In[277]:= ParametricPlot3D[{r Cos[t], r Sin[t], g[r]}, {r, 0, π},
            {t, 0, 2π}, PlotPoints- > {30, 30}]
```

□

We now demonstrate a volume problem that requires the method of disks.

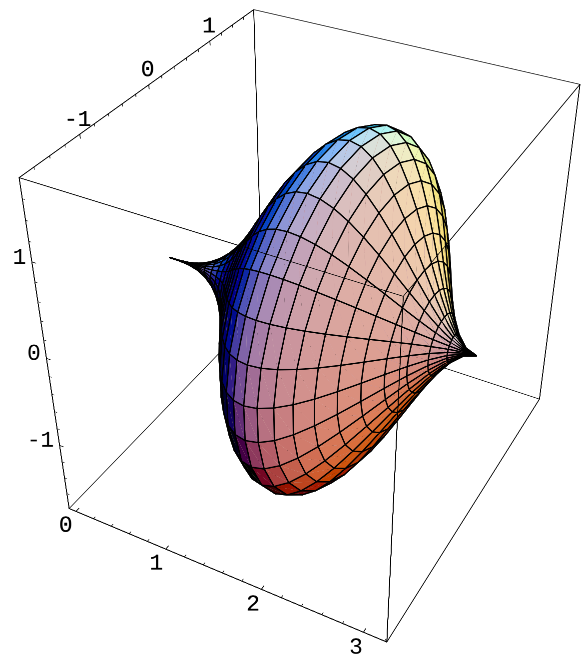


Figure 2.24: $g(x)$ revolved about the x-axis

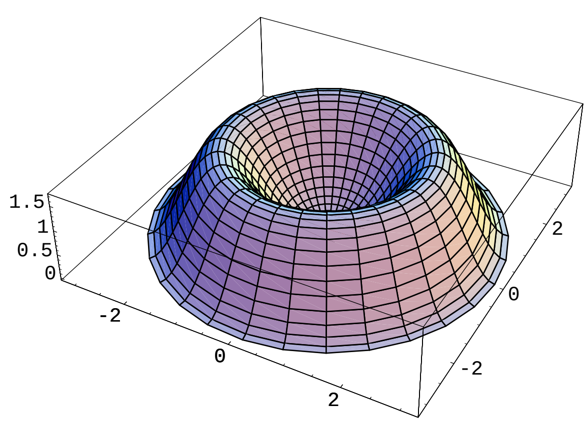


Figure 2.25: $g(x)$ revolved about the y-axis

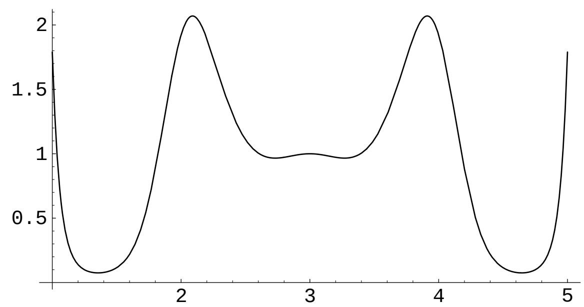


Figure 2.26: $f(x)$ for $1 \le x \le 5$

Example 51. Let

$$f(x) = e^{-(x-3)\cos[4(x-3)]}.$$

Approximate the volume of the solid obtained by revolving the region bounded by the graphs of $y = f(x)$, $x = 1$, $x = 5$, and the x-axis about the x-axis.

Solution. Proceeding as in the previous example, we first define and graph f on the interval $[1, 5]$ in Figure 2.26.

```
In[278]:= f[x_] = Exp[-(x - 3)^2 Cos[4(x - 3)]];
          Plot[f[x], {x, 1, 5}, AspectRatio- > Automatic]
```

In this case, an approximation is desired so we use `NIntegrate` to approximate the integral $V = \int_1^5 \pi\,[f(x)]^2\,dx$.

```
In[279]:= NIntegrate[π f[x]^2, {x, 1, 5}]
Out[279]= 16.0762
```

In the same manner as before, `ParametricPlot3D` can be used to visualize the resulting solid by graphing the set of equations given parametrically by

$$\begin{cases} x = r \\ y = f(r)\cos t \\ z = f(r)\sin t \end{cases}$$

for r between 1 and 5 and t between 0 and 2π. In this case, polar coordinates are used in the y,z-plane with the distance from the x-axis given by $f(x)$. Because r replaces x in the new coordinate system, $f(x)$ becomes $f(r)$ in these equations. See Figure 2.27

```
In[280]:= ParametricPlot3D[{r, f[r] Cos[t], f[r] Sin[t]},
            {r, 1, 5}, {t, 0, 2π}, PlotPoints- > {45, 35}]
```

□

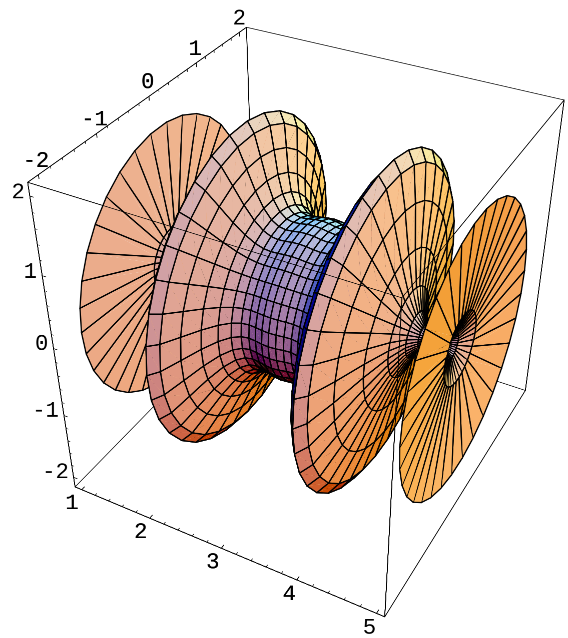


Figure 2.27: $f(x)$ revolved about the x-axis

Surface Area

Let $y = f(x)$ be a non-negative function for which $f'(x)$ is continuous on an interval $[a, b]$. Then the **surface area** of the solid of revolution obtained by revolving the region bounded by the graphs of $y = f(x)$, $x = a$, $x = b$, and the x-axis about the x-axis is given by

$$SA = 2\pi \int_a^b f(x)\sqrt{1 + \left[f'(x)\right]^2}\,dx. \tag{2.18}$$

Example 52 (Gabriel's Horn). Gabriel's Horn is the solid of revolution obtained by revolving the area of the region bounded by $y = 1/x$ and the x-axis for $x \geq 1$ about the x-axis. Show that the surface area of Gabriel's Horn is infinite but that its volume is finite.

Solution. After defining $f(x) = 1/x$, we use `ParametricPlot3D` to visualize a portion of Gabriel's Horn in Figure 2.28.

```
In[281]:= f[x_] = 1/x;
          ParametricPlot3D[{r, f[r] Cos[t], f[r] Sin[t]},
            {r, 1, 10}, {t, 0, 2π}, PlotPoints- > {40, 40},
            ViewPoint- > {-1.509,  -2.739,  1.294}]
```

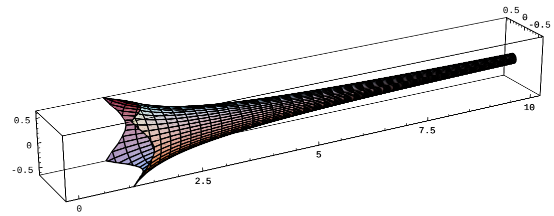


Figure 2.28: A portion of Gabriel's Horn

Using (2.18), the surface area of Gabriel's Horn is given by the improper integral

$$SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx = 2\pi \lim_{L\to\infty} \int_1^L \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx.$$

```
In[282]:= step1 = Integrate[2 π f[x] Sqrt[1 + f′[x]^2], {x, 1, capl}]
Integrate :: "gener" : "Unabletocheckconvergence"
Out[282]= -π (-√2 + ArcSinh[1]) +
          Sqrt[1 + 1/capl^4] π (-1 + capl^2 ArcSinh[capl^2]/Sqrt[1 + capl^4])

In[283]:= Limit[step1, capl-> ∞]
Out[283]= ∞
```

On the other hand, using (2.16) the volume of Gabriel's Horn is given by the improper integral

$$SA = 2\pi \int_1^\infty \frac{1}{x^2}\,dx = \pi \lim_{L\to\infty} \int_1^L \frac{1}{x^2}\,dx,$$

which converges to π.

```
In[284]:= step1 = Integrate[π f[x]^2, {x, 1, capl}]
Out[284]= π - π/capl

In[285]:= Limit[step1, capl-> ∞]
Out[285]= π

In[286]:= Integrate[π f[x]^2, {x, 1, ∞}]
Out[286]= π
```

□

2.4 Series

2.4.1 Introduction to Sequences and Series

Sequences and series are usually discussed in the third quarter or second semester of introductory calculus courses. Most students find that it is one of the most difficult topics covered in calculus. A **sequence** is a function with domain consisting of the positive integers. The **terms** of the sequence $\{a_n\}$ are $a_1, a_2, a_3, \ldots$. The nth term is a_n; the $(n+1)$st term is a_{n+1}. If $\lim_{n\to\infty} a_n = L$, we say that $\{a_n\}$ **converges** to L. If $\{a_n\}$ does not converge, $\{a_n\}$ **diverges**. We can often prove that a sequence converges by applying the following theorem.

Theorem 6. *Every bounded monotonic sequence converges.*

A sequence $\{a_n\}$ is monotonic if $\{a_n\}$ is increasing ($a_{n+1} \geq a_n$ for all n) or decreasing ($a_{n+1} \leq a_n$ for all n).

In particular, Theorem 6 gives us the following special cases.

1. If $\{a_n\}$ has positive terms and is eventually decreasing, $\{a_n\}$ converges.
2. If $\{a_n\}$ has negative terms and is eventually increasing $\{a_n\}$ converges.

After you have defined a sequence, use `Table` to compute the first few terms of the sequence.

1. `Table[a[n],{n,1,m}]` returns $a_1, a_2, a_3, \ldots, a_m$.
2. `Table[a[n],{n,k,m}]` returns $a_k, a_{k+1}, a_{k+2}, \ldots, a_m$.

Example 53. Let

$$a_n = \frac{50^n}{n!}.$$

Show that $\lim_{n\to\infty} a_n = 0$.

Solution. We remark that the symbol $n!$ in the denominator of a_n represents the **factorial sequence**:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1.$$

We begin by defining a_n and then computing the first few terms of the sequence with `Table`.

```
In[287]:= a[n_] = 50^n/n!;
          afewterms = Table[a[n], {n, 1, 10}]

          N[afewterms]
```

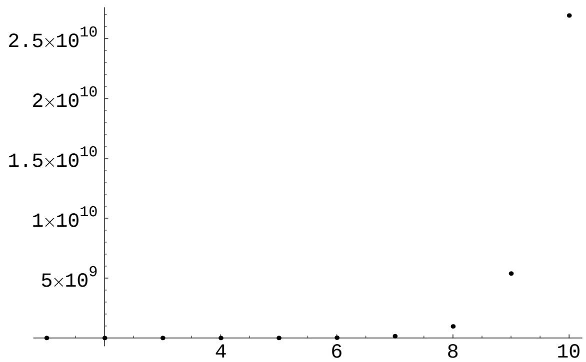


Figure 2.29: The first few terms of a_n

Out[287]= $\left\{50, 1250, \frac{62500}{3}, \frac{781250}{3}, \frac{7812500}{3}, \frac{195312500}{9}, \frac{9765625000}{63}, \frac{61035156250}{63}, \frac{3051757812500}{567}, \frac{15258789062500}{567}\right\}$

Out[287]= $\{50., 1250., 20833.3, 260417., 2.60417\, 10^6, 2.17014\, 10^7, 1.5501\, 10^8, 9.68812\, 10^8, 5.38229\, 10^9, 2.69114\, 10^{10}\}$

The first few terms increase in magnitude. In fact, this is further confirmed by graphing the first few terms of the sequence with `ListPlot` in Figure 2.29. Based on the graph and the values of the first few terms we might incorrectly conclude that the sequence diverges.

```
In[288]:= ListPlot[afewterms]
```

However, notice that

$$a_{n+1} = \frac{50}{n+1} a_n \Rightarrow \frac{a_{n+1}}{a_n} = \frac{50}{n+1}.$$

Because $50/(n+1) < 1$ for $n > 49$, we conclude that the sequence is decreasing for $n > 49$. Because it has positive terms, it is bounded below by 0 so the sequence converges by Theorem 6. Let $L = \lim_{n\to\infty} a_n$. Then,

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \frac{50}{n+1} a_n \\ L &= \lim_{n\to\infty} \frac{50}{n+1} \cdot L \\ L &= 0. \end{aligned}$$

When we graph a larger number of terms, it is clear that the limit is 0. (See Figure 2.30.) In fact, it is a good exercise to show that for any real value of x, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

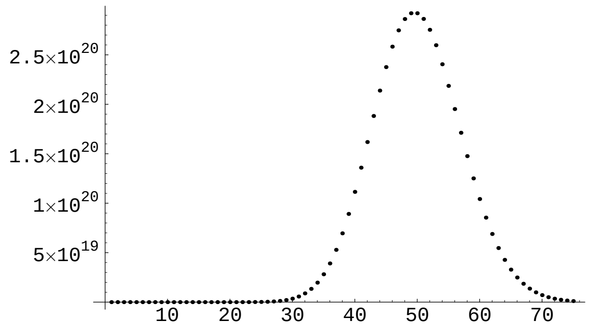


Figure 2.30: The first 75 terms of a_n

```
In[289]:= ListPlot[Evaluate[Table[a[k], {k, 1, 75}]]]
```

□

An **infinite series** is a series of the form

$$\sum_{k=1}^{\infty} a_k \tag{2.19}$$

where $\{a_n\}$ is a sequence. The nth **partial sum** of (2.19) is

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n. \tag{2.20}$$

Notice that the partial sums of the series (2.19) form a sequence $\{s_n\}$. Hence, we say that the infinite series (2.19) **converges** to L if the sequence of partial sums $\{s_n\}$ converges to L and write

$$\sum_{k=1}^{\infty} a_k = L.$$

The infinite series (2.19) **diverges** if the sequence of partial sums diverges. Given the infinite series (2.19),

```
Sum[a[k],{k,1,n}]
```

calculates the nth partial sum (2.20). In *some* cases, if the infinite series (2.19) converges,

```
Sum[a[k],{k,1,Infinity}]
```

can compute the value of the infinite sum. You should think of the `Sum` function as a ”fragile” command and be certain to carefully examine its results.

Example 54. Determine whether each series converges or diverges. If the series converges, find its sum. (a) $\sum_{k=1}^{\infty}(-1)^{k+1}$ (b) $\sum_{k=2}^{\infty}\frac{2}{k^2-1}$ (c) $\sum_{k=0}^{\infty} ar^k$

Solution. For (a), we compute the nth partial sum (2.20) in `sn` with `Sum`.

```
In[290]:= sn = Sum[(-1)^(k + 1), {k, 1, n}]
Out[290]= 1/2 (1 - (-1)^n)
```

Notice that the odd partial sums are 1:

$$s_{2n+1} = \frac{1}{2}\left((-1)^{2n+1+1} + 1\right) = \frac{1}{2}(1 + 1) = 1$$

while the even partial sums are 0:

$$s_{2n} = \frac{1}{2}\left((-1)^{2n+1} + 1\right) = \frac{1}{2}(-1 + 1) = 0.$$

We confirm that the limit of the partial sums does not exist with `limit`. Mathematica’s result indicates that it cannot determine the limit.

```
In[291]:= Limit[sn, n- > ∞]
Out[291]= Limit[1/2 (1 - (-1)^n), n → ∞]
```

However, when we attempt to compute the infinite sum with `Sum`, Mathematica is able to determine that the sum diverges.

```
In[292]:= Sum[(-1)^(k + 1), {k, 1, ∞}]
Sum :: "div" : "Sumdoesnotconverge."
Out[292]= Sum_{k=1}^{∞} (-1)^(k+1)
```

Thus, the series diverges.

For (b), we have a *telescoping series*. Using partial fractions,

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{2}{k^2-1} &= \sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)\\ &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \ldots\end{aligned}$$

we see that the nth partial sum is given by

$$s_n = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$$

and $s_n \to 3/2$ as $n \to \infty$ so the series converges to $3/2$:

$$\sum_{k=2}^{\infty}\frac{2}{k^2-1} = \frac{3}{2}.$$

We perform the same steps with Mathematica using `Sum` and `Limit`.

```
In[293]:= sn = Sum[1/(k - 1) - 1/(k + 1), {k, 2, n}]
```

$$\text{Out[293]=}\ \frac{(-1+n)\ (2+3\ n)}{2\ n\ (1+n)}$$

```
In[294]:= Apart[sn]
```

$$\text{Out[294]=}\ \frac{3}{2}-\frac{1}{n}-\frac{1}{1+n}$$

```
In[295]:= Limit[sn, n-> ∞]
```

$$\text{Out[295]=}\ \frac{3}{2}$$

(c) A series of the form $\sum_{k=0}^{\infty} ar^k$ is called a **geometric series**. We compute the nth partial sum of the geometric series with `Sum`.

```
In[296]:= sn = Sum[a r^k, {k, 0, n}]
```

$$\text{Out[296]=}\ \frac{a\ (-1+r^{1+n})}{-1+r}$$

When using `Limit` to determine the limit of s_n as $n \to \infty$, we see that Mathematica returns the limit unevaluated because Mathematica does not know the value of r.

```
In[297]:= Limit[sn, n-> ∞]
```

$$\text{Out[297]=}\ \text{Limit}\left[\frac{a\ (-1+r^{1+n})}{-1+r}, n\to\infty\right]$$

In fact, the geometric series diverges if $|r| \geq 1$ and converges if $|r| < 1$. Observe that if we simply compute the sum with `sum`, Mathematica returns $a/(1-r)$ which is correct if $|r| < 1$ but incorrect if $|r| \geq 1$.

```
In[298]:= Sum[a r^k, {k, 0, ∞}]
```

$$\text{Out[298]=}\ \frac{a}{1-r}$$

However, the result of entering

```
In[299]:= Sum[(-5/3)^k, {k, 0, ∞}]
Sum :: "div" :  "Sumdoesnotconverge."
```

$$\text{Out[299]=}\ \sum_{k=0}^{\infty}\left(-\frac{5}{3}\right)^k$$

is correct because the series $\sum_{k=0}^{\infty}\left(-\frac{5}{3}\right)^k$ is geometric with $|r| = 5/3 \geq 1$ and, consequently, diverges. Similarly,

```
In[300]:= Sum[9 (1/10)^k, {k, 1, ∞}]
Out[300]= 1
```

is correct because $\sum_{k=1}^{\infty} 9\left(\frac{1}{10}\right)^k$ is geometric with $a = 9/10$ and $r = 1/10$ so the series converges to

$$\frac{a}{1-r} = \frac{9/10}{1-1/10} = 1.$$

□

2.4.2 Convergence Tests

Frequently used convergence tests are stated in the following theorems.

Theorem 7 (The Divergence Test). *Let $\sum_{k=1}^{\infty} a_k$ be an infinite series. If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.*

Theorem 8 (The Integral Test). *Let $\sum_{k=1}^{\infty} a_k$ be an infinite series with positive terms. If $f(x)$ is a decreasing continuous function for which $f(k) = a_k$ for all k, then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.*

Theorem 9 (The Ratio Test). *Let $\sum_{k=1}^{\infty} a_k$ be an infinite series with positive terms and let $\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.*

1. *If $\rho < 1$, $\sum_{k=1}^{\infty} a_k$ converges.*

2. *If $\rho > 1$, $\sum_{k=1}^{\infty} a_k$ diverges.*

3. *If $\rho = 1$, the Ratio Test is inconclusive.*

Theorem 10 (The Root Test). *Let $\sum_{k=1}^{\infty} a_k$ be an infinite series with positive terms and let $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$.*

1. *If $\rho < 1$, $\sum_{k=1}^{\infty} a_k$ converges.*

2. *If $\rho > 1$, $\sum_{k=1}^{\infty} a_k$ diverges.*

3. *If $\rho = 1$, the Root Test is inconclusive.*

Theorem 11 (The Limit Comparison Test). *Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be infinite series with positive terms and let $L = \lim_{k\to\infty} \frac{a_k}{b_k}$. If $0 < L < \infty$, then either both series converge or both series diverge.*

Example 55. Determine whether each series converges or diverges. (a) $\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^k$ (b) $\sum_{k=1}^{\infty} \frac{1}{k^p}$ (c) $\sum_{k=1}^{\infty} \frac{k}{3^k}$ (d) $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ (e) $\sum_{k=1}^{\infty} \left(\frac{k}{4k+1}\right)^k$ (f) $\sum_{k=1}^{\infty} \frac{2\sqrt{k}+1}{(\sqrt{k}+1)(2k+1)}$

Solution. (a) Using `Limit`, we see that the limit of the terms is $e \neq 0$ so the series diverges by the the Divergence test, Theorem 7.

```
In[301]:= Limit[(1 + 1/k)^k, k- > ∞]
Out[301]= e
```

It is a very good exercise to show that the limit of the terms of the series is e by hand. Let $L = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k$. Take the logarithm of each side of this equation and apply

L'Hôpital's rule:

$$\begin{aligned}
\ln L &= \lim_{k\to\infty} \ln\left(1+\frac{1}{k}\right)^k \\
\ln L &= \lim_{k\to\infty} k\ln\left(1+\frac{1}{k}\right) \\
\ln L &= \lim_{k\to\infty} \frac{\ln\left(1+\frac{1}{k}\right)}{\frac{1}{k}} \\
\ln L &= \lim_{k\to\infty} \frac{\frac{1}{1+\frac{1}{k}}\cdot -\frac{1}{k^2}}{-\frac{1}{k^2}} \\
\ln L &= 1.
\end{aligned}$$

Exponentiating yields $L = e^{\ln L} = e^1 = e$. (b) A series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a p-**series**. Let $f(x) = x^{-p}$. Then, $f(x)$ is continuous and decreasing for $x \geq 1$ and $f(k) = k^{-p}$. Then,

$$\int_1^\infty x^{-p} dx = \begin{cases} \infty, \text{if } p \leq 1 \\ 1/(p-1), \text{if } p > 1 \end{cases}$$

so the p-series converges if $p > 1$ and diverges if $p \leq 1$. If $p = 1$, the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is called the **harmonic series**.

```
In[302]:= s1 = Integrate[x^(-p), {x, 1, ∞}]
Out[302]= If[Re[p] > 1, 1/(-1 + p), ∫_1^∞ x^-p dx]
```

(c) Let $f(x) = x \cdot 3^{-x}$. Then, $f(k) = k \cdot 3^{-k}$ and $f(x)$ is decreasing for $x > 1/\ln 3$.

```
In[303]:= f[x_] = x 3^(-x);
          Factor[f'[x]]
Out[303]= -3^-x (-1 + x Log[3])

In[304]:= Solve[-1 + x Log[3] == 0]
Out[304]= {{x → 1/Log[3]}}
```

Using `Integrate`, we see that the improper integral $\int_1^\infty f(x)\, dx$ converges.

```
In[305]:= ival = Integrate[f[x], {x, 1, ∞}]

          N[ival]
Out[305]= (1 + Log[3])/(3 Log[3]^2)
Out[305]= 0.579592
```

Thus, by the Integral test, Theorem 8, we conclude that the series converges. Note that when applying the Integral test, if the improper integral converges its value is *not* the value of the sum of the series. In this case, we see that Mathematica is able to evaluate the sum with `Sum` and the series converges to 3/4.

```
In[306]:= Sum[k 3^(-k), {k, 1, ∞}]
```

```
Out[306]= 3/4
```

(d) If a_k contains factorials, the Ratio test is a good first test to try. After defining a_k we compute

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to\infty}\frac{\frac{[(k+1)!]^2}{[2(k+1)]}}{\frac{(k!)^2}{(2k)!}}$$
$$=\lim_{k\to\infty}\frac{(k+1)!\cdot(k+1)!}{k!\cdot k!}\frac{(2k)!}{(2k+2)!}$$
$$=\lim_{k\to\infty}\frac{(k+1)^2}{(2k+2)(2k+1)}=\lim_{k\to\infty}\frac{(k+1)}{2(2k+1)}=\frac{1}{4}.$$

Because $1/4 < 1$, the series converges by the Ratio test. We confirm these results with Mathematica.

Remark. Use `FullSimplify` instead of `Simplify` to simplify expressions involving factorials.

```
In[307]:= a[k_] = (k!)^2/(2k)!;
          s1 = FullSimplify[a[k + 1]/a[k]]
Out[307]= (1 + k)/(2 + 4 k)
In[308]:= Limit[s1, k- > ∞]
Out[308]= 1/4
```

We illustrate that we can approximate the sum using `N` and `Sum` as follows.

```
In[309]:= ev = Sum[a[k], {k, 1, ∞}]
Out[309]= 1/27 (9 + 2 √3 π)
In[310]:= N[ev]
Out[310]= 0.7364
```

(e) Because

$$\lim_{k\to\infty}\sqrt[k]{\left(\frac{k}{4k+1}\right)^k}=\lim_{k\to\infty}\frac{k}{4k+1}=\frac{1}{4}<1,$$

the series converges by the Root test.

```
In[311]:= a[k_] = (k/(4k + 1))^k;
          Limit[a[k]^(1/k), k- > ∞]
Out[311]= 1/4
```

As with (d), we can approximate the sum with `N` and `Sum`.

```
In[312]:= ev = Sum[a[k], {k, 1, ∞}]
Out[312]= Sum_{k=1}^{∞} a[k]
In[313]:= N[ev]
```

```
Out[313]= 0.265757
```

(f) We use the Limit Comparison test and compare the series to $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k}$, which diverges because it is a p-series with $p = 1$. Because

$$0 < \lim_{k\to\infty} \frac{\frac{2\sqrt{k}+1}{(\sqrt{k}+1)(2k+1)}}{\frac{1}{k}} = 1 < \infty$$

and the harmonic series diverges, the series diverges by the Limit Comparison test.

```
In[314]:= a[k_] = (2Sqrt[k] + 1)/((Sqrt[k] + 1)(2k + 1));
          b[k_] = 1/k;
          Limit[a[k]/b[k], k- > ∞]
Out[314]= 1
```

□

2.4.3 Alternating Series

An **alternating series** is a series of the form

$$\sum_{k=1}^{\infty}(-1)^k a_k \qquad \text{or} \qquad \sum_{k=1}^{\infty}(-1)^{k+1} a_k \tag{2.21}$$

where $\{a_k\}$ is a sequence with positive terms.

Theorem 12 (Alternating Series Test). *If $\{a_k\}$ is decreasing and $\lim_{k\to\infty} a_k = 0$, the alternating series* (2.21) *converges.*

The alternating series (2.21) **converges absolutely** if $\sum_{k=1}^{\infty} a_k$ converges.

Theorem 13. *If the alternating series* (2.21) *converges absolutely, it converges.*

If the alternating series (2.21) converges but does not converge absolutely, we say that it **conditionally converges**.

Example 56. Determine whether each series converges or diverges. If the series converges, determine whether the convergence is conditional or absolute. (a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ (b) $\sum_{k=1}^{\infty}(-1)^{k+1} \frac{(k+2)!}{4^k(k!)^2}$ (c) $\sum_{k=1}^{\infty}(-1)^{k+1}\left(1+\frac{1}{k}\right)^k$

Solution. (a) Because $\{1/k\}$ is decreasing and $1/k \to 0$ as $k \to \infty$, the series converges. The series does not converge absolutely because the harmonic series diverges. Hence, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$, which is called the **alternating harmonic series**, converges conditionally. We see that this series converges to $\ln 2$ with `Sum`.

```
In[315]:= a[k_] = (-1)^(k + 1)/k;
          Sum[a[k], {k, 1, ∞}]
```

```
Out[315]= Log[2]
```

(b) We test for absolute convergence first using the Ratio test. Because

$$\lim_{k\to\infty} \frac{\frac{((k+1)+2)!}{4^{(}k+1)[(k+1)!]^2}}{\frac{(k+2)!}{4^k(k!)^2}} = \lim_{k\to\infty} \frac{k+2}{4(k+1)^2} = 0 < 1,$$

```
In[316]:= a[k_] = (k + 1)!/(4^k (k!)^2);
          s1 = FullSimplify[a[k + 1]/a[k]]

          Limit[s1, k- > ∞]
              2 + k
Out[316]= ------------
          4 (1 + k)^2
Out[316]= 0
```

the series converges absolutely by the Ratio test. Absolute convergence implies convergence so the series converge. (c) Because $\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e$, $\lim_{k\to\infty} (-1)^{k+1} \left(1+\frac{1}{k}\right)^k$ does not exist, so the series diverges by the Divergence test. We confirm that the limit of the terms is not zero with `Limit`.

```
In[317]:= a[k_] = (-1)^(k + 1)(1 + 1/k)^k;
          Sum[a[k], {k, 1, ∞}]
Sum :: "div" : "Sumdoesnotconverge."
           ∞
Out[317]=  ∑ a[k]
          k=1
In[318]:= Limit[a[k], k- > ∞]
Out[318]= Limit[(-1)^(1+k) (1 + 1/k)^k, k → ∞]
```

□

2.4.4 Power Series

Let x_0 be a number. A power series in $x - x_0$ is a series of the form

$$\sum_{k=0}^{\infty} a_k \left(x - x_0\right)^k. \tag{2.22}$$

A fundamental problem is determining the values of x, if any, for which the power series converges.

Theorem 14. *For the power series* (2.22), *exactly one of the following is true.*

1. *The power series converges absolutely for all values of* x. *The interval of convergence is* $(-\infty, \infty)$.

2. *There is a positive number r so that the series converges absolutely if* $x_0 - r < x < x_0 + r$. *The series may or may not converge at* $x = x_0 - r$ *and* $x = x_0 + r$. *The interval of convergence will be one of* $(x_0 - r, x_0 + r)$, $[x_0 - r, x_0 + r)$, $(x_0 - r, x_0 + r]$, *or* $[x_0 - r, x_0 + r]$.

3. *The series converges only if* $x = x_0$. *The interval of convergence is* $\{x_0\}$.

Example 57. Determine the interval of convergence for each of the following power series. (a) $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ (b) $\sum_{k=0}^{\infty} \frac{k!}{1000^k} (x-1)^k$ (c) $\sum_{k=1}^{\infty} \frac{2^k}{\sqrt{k}} (x-4)^k$

Solution. (a) We test for absolute convergence first using the Ratio test. Because

$$\lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1}}{(2(k+1)+1)!} x^{2(k+1)+1}}{\frac{(-1)^k}{(2k+1)!} x^{2k+1}} \right| = \lim_{k\to\infty} \frac{1}{2(k+1)(2k+3)} x^2 = 0 < 1$$

```
In[319]:= a[x_, k_] = (-1)^k /(2k + 1)!x^(2k + 1);
          s1 = FullSimplify[a[x, k + 1]/a[x, k]]

          Limit[s1, k- > ∞]
Out[319]= -x^2/(6 + 10 k + 4 k^2)
Out[319]= 0
```

for all values of x, we conclude that the series converges absolutely for all values of x; the interval of convergence is $(-\infty, \infty)$. In fact, we will see later that this series converges to $\sin x$:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots,$$

which means that the partial sums of the series converge to $\sin x$. Graphically, we can visualize this by graphing partial sums of the series together with the graph of $y = \sin x$. Note that the partial sums of a series are a recursively defined function: $s_n = s_{n-1} + a_n$, $s_0 = a_0$. We use this observation to define `p` to be the nth partial sum of the series. We use the form `p[x_,n_]:=p[x,n]=...` so that Mathematica "remembers" the partial sums computed. That is, once `p[x,3]` is computed, Mathematica need not recompute `p[x,3]` when computing `p[x,4]`.

```
In[320]:= Clear[p]

          p[x_, 0] = a[0];
          p[x_, n_] := p[x, n] = p[x, n - 1] + a[x, n]
In[321]:= p[x, 2]
Out[321]= x - x^3/6 + x^5/120
```

In Figure 2.31 we graph $p_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ together with $y = \sin x$ for $n = 1$, 5, and 10. In the graphs, notice that as n increases, the graphs of $p_n(x)$ more closely resemble the graph of $y = \sin x$.

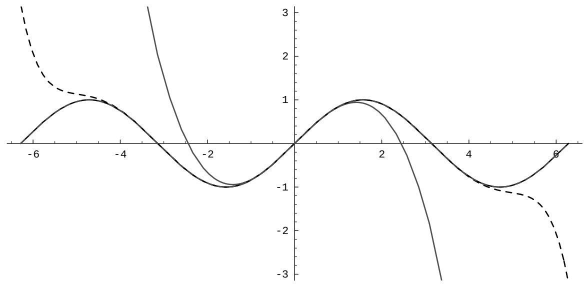


Figure 2.31: $y = \sin x$ together with the graphs of $p_1(x)$, $p_5(x)$, and $p_{10}(x)$

```
In[322]:= Plot[{Sin[x], p[x, 1], p[x, 5], p[x, 10]}, {x, -2π, 2π},
            PlotRange- > {-π, π}, AspectRatio- > Automatic,
            PlotStyle- > {GrayLevel[0], GrayLevel[0.3],
                Dashing[{0.01}], {GrayLevel[0.3], Dashing[{0.01}]}}]
```

(b) As in (a), we test for absolute convergence first using the Ratio test. Because

$$\lim_{k\to\infty}\left|\frac{\frac{(k+1)k!}{1000^{k+1}}(x-1)^{k+1}}{\frac{k!}{1000^k}(x-1)^k}\right| = \frac{1}{1000}(k+1)|x-1| = \begin{cases}0, \text{if } x = 1\\ \infty, \text{if } x \neq 1\end{cases}.$$

```
In[323]:= a[x_, k_] = k!/1000^k (x - 1)^k;
          s1 = FullSimplify[a[x, k + 1]/a[x, k]]

          Limit[s1, k- > ∞]
Out[323]= (1 + k) (-1 + x)
          ----------------
                1000
Out[323]= Indeterminate
```

Be careful of your interpretation of the result of the `Limit` command because Mathematica does not consider the case $x = 1$ separately: if $x = 1$ the limit is 0. Because $0 < 1$ the series converges by the Ratio test.

The series converges only if $x = 1$; the interval of convergence is $\{1\}$. You should observe that if you graph several partial sums for ”small” values of n, you might incorrectly conclude that the series converges. (c) Use the Ratio test to check absolute convergence first:

$$\lim_{k\to\infty}\left|\frac{\frac{2^{k+1}}{\sqrt{k+1}}(x-4)^{k+1}}{\frac{2^k}{\sqrt{k}}(x-4)^k}\right| = \lim_{k\to\infty} 2\sqrt{\frac{k}{k+1}}|x-4| = 2|x-4|.$$

By the Ratio test, the series converges absolutely if $2|x-4| < 1$. We solve this inequality for x with `InequalitySolve` to see that $2|x-4| < 1$ if $7/2 < x < 9/2$.

```
In[324]:= a[x_, k_] = 2^k /Sqrt[k] (x - 4)^k;
          s1 = FullSimplify[Abs[a[x, k + 1]/a[x, k]]]

          Limit[s1, k- > ∞]
```

```
Out[324]= 2 Abs[Sqrt[k/(1 + k)] (-4 + x)]
Out[324]= 2 Abs[-4 + x]

In[325]:= << Algebra`InequalitySolve`

          InequalitySolve[2 Abs[x - 4] < 1, x]
Out[325]= 7/2 < x < 9/2
```

We check $x = 7/2$ and $x = 9/2$ separately. If $x = 7/2$, the series becomes $\sum_{k=1}^{\infty}(-1)^k \frac{1}{\sqrt{k}}$, which converges conditionally.

```
In[326]:= Simplify[a[x, k]/.x-> 7/2]
Out[326]= (-1)^k/Sqrt[k]
```

On the other hand, if $x = 9/2$, the series is $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, which diverges. We conclude that the interval of convergence is $[7/2, 9/2)$.

```
In[327]:= Simplify[a[x, k]/.x-> 9/2]
Out[327]= 1/Sqrt[k]
```

□

2.4.5 Taylor and Maclaurin Series

Let $y = f(x)$ be a function with derivatives of all orders at $x = x_0$. The **Taylor series** for $f(x)$ about $x = x_0$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{2.23}$$

The **Maclaurin series** for $f(x)$ is the Taylor series for $f(x)$ about $x = 0$. If $y = f(x)$ has derivatives up to at least order n at $x = x_0$, the nth degree **Taylor polynomial** for $f(x)$ about $x = x_0$ is

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{2.24}$$

The nth degree **Maclaurin polynomial** for $f(x)$ is the nth degree Taylor polynomial for $f(x)$ about $x = 0$. Generally, finding Taylor and Maclaurin series using the definition is a tedious task at best.

Example 58. Find the first few terms of (a) the Maclaurin series and (b) the Taylor series about $x = \pi/4$ for $f(x) = \tan x$.

Solution. (a) After defining $f(x) = \tan x$, we use `Table` together with `/.` and `D` to compute $f^{(k)}(0)/k!$ for $k = 0, 1, \ldots, 8$.

```
In[328]:= f[x_] = Tan[x];
          Table[
           {k, D[f[x], {x, k}], D[f[x], {x, k}]/.x-> 0}, {k, 0, 8}]
Out[328]= {{0, Tan[x], 0}, {1, Sec[x]^2, 1}, {2, 2 Sec[x]^2 Tan[x], 0},
           {3, 2 Sec[x]^4 + 4 Sec[x]^2 Tan[x]^2, 2},
           {4, 16 Sec[x]^4 Tan[x] + 8 Sec[x]^2 Tan[x]^3, 0},
           {5, 16 Sec[x]^6 + 88 Sec[x]^4 Tan[x]^2 + 16 Sec[x]^2 Tan[x]^4, 16},
           {6, 272 Sec[x]^6 Tan[x] +
              416 Sec[x]^4 Tan[x]^3 + 32 Sec[x]^2 Tan[x]^5, 0},
           {7, 272 Sec[x]^8 + 2880 Sec[x]^6 Tan[x]^2 +
              1824 Sec[x]^4 Tan[x]^4 + 64 Sec[x]^2 Tan[x]^6, 272},
           {8, 7936 Sec[x]^8 Tan[x] + 24576 Sec[x]^6 Tan[x]^3 +
              7680 Sec[x]^4 Tan[x]^5 + 128 Sec[x]^2 Tan[x]^7,
            0}}
```

Using the values in the table, we apply the definition to see that the Maclaurin series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \ldots$$

For (b), we repeat (a) using $x = \pi/4$ instead of $x = 0$

```
In[329]:= f[x_] = Tan[x];
          Table[
           {k, D[f[x], {x, k}], D[f[x], {x, k}]/.x-> π/4}, {k, 0, 8}]
Out[329]= {{0, Tan[x], 1}, {1, Sec[x]^2, 2}, {2, 2 Sec[x]^2 Tan[x], 4},
           {3, 2 Sec[x]^4 + 4 Sec[x]^2 Tan[x]^2, 16},
           {4, 16 Sec[x]^4 Tan[x] + 8 Sec[x]^2 Tan[x]^3, 80},
           {5, 16 Sec[x]^6 + 88 Sec[x]^4 Tan[x]^2 + 16 Sec[x]^2 Tan[x]^4, 512},
           {6, 272 Sec[x]^6 Tan[x] +
              416 Sec[x]^4 Tan[x]^3 + 32 Sec[x]^2 Tan[x]^5, 3904},
           {7, 272 Sec[x]^8 + 2880 Sec[x]^6 Tan[x]^2 +
              1824 Sec[x]^4 Tan[x]^4 + 64 Sec[x]^2 Tan[x]^6, 34816},
           {8, 7936 Sec[x]^8 Tan[x] + 24576 Sec[x]^6 Tan[x]^3 +
              7680 Sec[x]^4 Tan[x]^5 + 128 Sec[x]^2 Tan[x]^7,
            354560}}
```

and then apply the definition to see that the Taylor series about $x = \pi/4$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4 + \frac{64}{15}\left(x - \frac{\pi}{4}\right)^5 + \frac{244}{45}\left(x - \frac{\pi}{4}\right)^6 + \ldots$$

From the series, we can see various Taylor and Maclaurin polynomials. For example, the third Maclaurin polynomial is

$$p_3(x) = x + \frac{1}{3}x^3$$

and the 4th degree Taylor polynomial about $x = \pi/4$ is

$$p_4(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4.$$

□

The command

```
Series[f[x],{x,x0,n}]
```

computes (2.23) to (at least) order $n - 1$. Because of the O-term in the result that represents the terms that are omitted from the power series for $f(x)$ expanded about the point $x = x_0$, the result of entering a `series` command is not a function that can be evaluated if x is a particular number. We remove the remainder (O-) term of the power series `Series[f[x],{x,x0,n}]` with the command `Normal` and can then evaluate the resulting polynomial for particular values of x.

Example 59. Find the first few terms of the Taylor series for $f(x)$ about $x = x_0$. (a) $f(x) = \cos x$, $x = 0$ (b) $f(x) = 1/x^2$, $x = 1$

Solution. Entering

```
In[330]:= Series[Cos[x], {x, 0, 4}]
```

Out[330]= $1 - \frac{x^2}{2} + \frac{x^4}{24} + O[x]^5$

computes the Maclaurin series to order 4. Entering

```
In[331]:= Series[Cos[x], {x, 0, 14}]
```

Out[331]= $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \frac{x^{12}}{479001600} - \frac{x^{14}}{87178291200} + O[x]^{15}$

computes the Maclaurin series to order 14. In this case, the Maclaurin series for $\cos x$ converges to $\cos x$ for all real x. To graphically see this, we define the function `p`. Given n, `p[n]` returns the Maclaurin polynomial of degree n for $\cos x$.

```
In[332]:= p[n_] := Series[Cos[x], {x, 0, n}]//Normal
```

```
In[333]:= p[8]
```

Out[333]= $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320}$

We then graph $\cos x$ together with the Maclaurin polynomial of degree $n = 2$, 4, 8 and 16 on the interval $[-3\pi/2, 3\pi/2]$ in Figure 2.32. Notice that as n increases, the graph of the Maclaurin polynomial more closely resembles the graph of $\cos x$. We would see the same pattern if we increased the length of the interval and the value of n.

```
In[334]:= somegraphs = Table[Plot[Evaluate[{Cos[x], p[2^n]}],
              {x, -3π/2, 3π/2}, PlotRange- > {-3π/2, 3π/2},
              AspectRatio- > Automatic,
              PlotStyle- > {GrayLevel[0], GrayLevel[0.3]},
              DisplayFunction- > Identity],
             {n, 1, 4}]
Out[334]= BoxData({-Graphics-, -Graphics-, -Graphics-, -Graphics-})

In[335]:= toshow = Partition[somegraphs, 2]
Out[335]= BoxData({{-Graphics-, -Graphics-}, {-Graphics-, -Graphics-}})

In[336]:= Show[GraphicsArray[toshow]]
```

(b) After defining $f(x) = 1/x^2$, we compute the first 10 terms of the Taylor series for $f(x)$ about $x = 1$ with `Series`.

```
In[337]:= f[x_] = 1/x^2;
          p10 = Series[f[x], {x, 1, 10}]
Out[337]= 1 - 2 (x - 1) + 3 (x - 1)^2 - 4 (x - 1)^3 + 5 (x - 1)^4 - 6 (x - 1)^5 +
            7 (x - 1)^6 - 8 (x - 1)^7 + 9 (x - 1)^8 - 10 (x - 1)^9 + 11 (x - 1)^10 +
            O[x - 1]^11
```

In this case, the pattern for the series is relatively easy to see: the Taylor series for $f(x)$ about $x = 1$ is

$$\sum_{k=0}^{\infty}(-1)^k(k+1)(x-1)^k.$$

This series converges absolutely if

$$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}(k+2)(x-1)^{k+1}}{(-1)^k(k+1)(x-1)^k}\right| = |x-1| < 1$$

or $0 < x < 2$. The series diverges if $x = 0$ and $x = 2$. In this case, the series converges to $f(x)$ on the interval $(0, 2)$.

```
In[338]:= a[x_, k_] = (-1)^k (k + 1) (x - 1)^k;
          s1 = FullSimplify[Abs[a[x, k + 1]/a[x, k]]]
Out[338]= Abs[((2 + k) (-1 + x))/(1 + k)]

In[339]:= s2 = Limit[s1, k- > ∞]
Out[339]= Abs[-1 + x]

In[340]:= << Algebra`InequalitySolve`

          InequalitySolve[s2 < 1, x]
Out[340]= 0 < x < 2
```

To see this, we graph $f(x)$ together with the Taylor polynomial for $f(x)$ about $x = 1$ of degree n for large n. Regardless of the size of n, the graphs of $f(x)$ and the Taylor

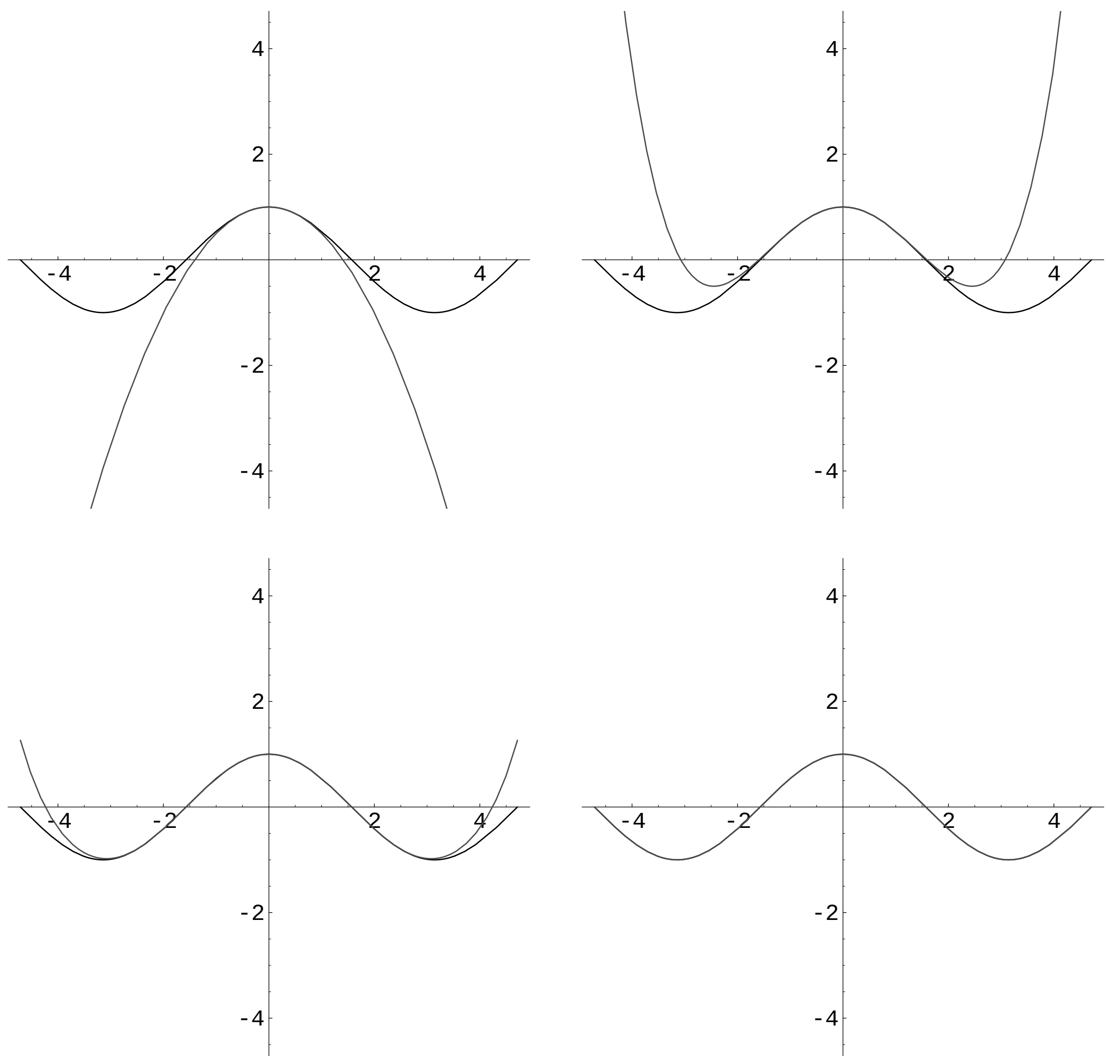


Figure 2.32: Graphs of $y = \cos x$ together with its second, fourth, eighth, and sixteenth Maclaurin polynomials

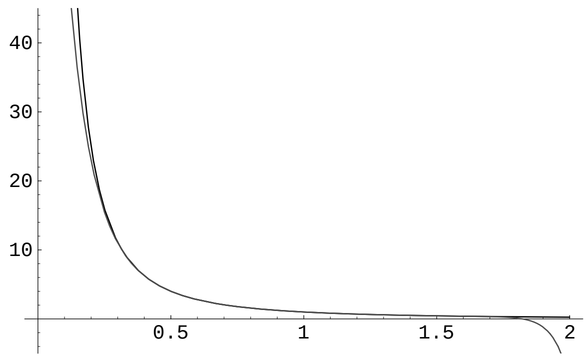


Figure 2.33: Graph of $f(x)$ together with the sixteenth degree Taylor polynomial about $x = 1$

polynomial closely resemble each other on the interval $(0, 2)$–but not at the endpoints or outside the interval. (See Figure 2.33.)

```
In[341]:= p[n_] := Series[f[x], {x, 1, n + 1}]//Normal

In[342]:= Plot[Evaluate[{f[x], p[16]}],
            {x, 0, 2}, PlotRange- > {-5, 45},
            PlotStyle- > {GrayLevel[0], GrayLevel[0.3]}]
```

□

2.4.6 Taylor's Theorem

Taylor's theorem states the relationship between $f(x)$ and the Taylor series for $f(x)$ about $x = x_0$.

Theorem 15 (Taylor's Theorem). *Let $y = f(x)$ have (at least) $n + 1$ derivatives on an interval I containing $x = x_0$. Then, for every number $x \in I$, there is a number z between x and x_0 so that*

$$f(x) = p_n(x) + R_n(x),$$

where $p_n(x)$ is given by (2.24) *and*

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}\left(x - x_0\right)^{n+1}. \tag{2.25}$$

Example 60. Use Taylor's theorem to show that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Solution. Let $f(x) = \sin x$. Then, for each value of x, there is a number z between 0 and x so that $\sin x = p_n(x) + R_n(x)$ where $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$ and $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$. Regardless of the value of n, $f^{(n+1)}(z)$ is one of $\sin z$, $-\sin z$, $\cos z$, or $-\cos z$, which are all bounded by 1. Then,

$$\left|\sin x - p_n(x)\right| = \left|\frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}\right|$$
$$\left|\sin x - p_n(x)\right| \le \frac{1}{(n+1)!} |x|^{n+1}$$

and $\frac{x^n}{n!} \to 0$ as $n \to \infty$ for all real values of x.

You should remember that the number z in $R_n(x)$ is guaranteed to exist by Taylor's theorem. However, from a practical point of view, you would rarely (if ever) need to compute the z value for a particular x value.

For illustrative purposes, we show the difficulties. Suppose we wish to approximate $\sin \pi/180$ using the Maclaurin polynomial of degree 4, $p_4(x) = x - \frac{1}{6}x^3$, for $\sin x$. The fourth remainder is

The Maclaurin polynomial of degree 4 for $\sin x$ is $\sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^4 = 0 + x + 0 \cdot x^2 + \frac{-1)(3!}{x}^3 + 0 \cdot x^4$.

$$R_4(x) = \frac{1}{120} \cos z \, x^5.$$

```
In[343]:= f[x_] = Sin[x];
          r5 = D[f[z], {z, 5}]/5! x^5
Out[343]= 1/120 x^5 Cos[z]
```

If $x = \pi/180$ there is a number z between 0 and $\pi/180$ so that

$$\left|R_4\left(\frac{\pi}{180}\right)\right| = \frac{1}{120} \cos z \left(\frac{\pi}{180}\right)^5$$
$$\le \frac{1}{120} \left(\frac{\pi}{180}\right)^5 \approx 0.135 \times 10^{-10},$$

which shows us that the maximum the error can be is $\frac{1}{120}\left(\frac{\pi}{180}\right)^5 \approx 0.135 \times 10^{-10}$.

```
In[344]:= maxerror = N[1/120 * (π/180)^5]
Out[344]= 1.3496 10^-11
```

Abstractly, the exact error can be computed. By Taylor's theorem, z satisfies

$$f\left(\frac{\pi}{180}\right) = p_4\left(\frac{\pi}{180}\right) + R_4\left(\frac{\pi}{180}\right)$$
$$\sin \frac{\pi}{180} = \frac{1}{180}\pi - \frac{1}{34992000}\pi^3 + \frac{1}{22674816000000}\pi^5 \cos z$$
$$0 = \frac{1}{180}\pi - \frac{1}{34992000}\pi^3 + \frac{1}{22674816000000}\pi^5 \cos z - \sin \frac{\pi}{180}.$$

We graph the right-hand side of this equation with `Plot` in Figure 2.34. The exact value of z is the z-coordinate of the point where the graph intersects the z-axis.

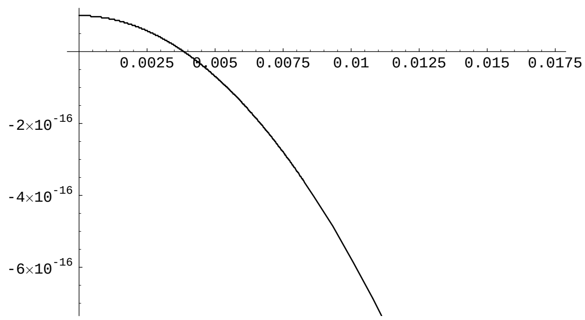


Figure 2.34: Finding z

```
In[345]:= p4 = Series[f[x], {x, 0, 4}]//Normal
```

Out[345]= $x - \frac{x^3}{6}$

```
In[346]:= exval = Sin[π/180]

          p4b = p4/.x- > π/180

          r5b = r5/.x- > π/180
```

Out[346]= $\text{Sin}\left[\frac{\pi}{180}\right]$

Out[346]= $\frac{\pi}{180} - \frac{\pi^3}{34992000}$

Out[346]= $\frac{\pi^5\ \text{Cos}[z]}{22674816000000}$

```
In[347]:= toplot = r5b + p4b - exval;
          Plot[toplot, {z, 0, π/180}]
```

We can use `FindRoot` to approximate z, if we increase the number of digits carried in floating point calculations with `WorkingPrecision`.

```
In[348]:= exz =
            FindRoot[toplot == 0, {z, 0.004}, WorkingPrecision- > 32]
Out[348]= {z → 0.00380861491655416079493316330124}
```

Alternatively, we can compute the exact value of z with `Solve`

```
In[349]:= cz = Solve[toplot == 0, z]
Solve :: "ifun" : "Inversefunctionsarebeingused
     bySolve, so some solutions may not be found."
```

Out[349]= $\left\{\left\{z \to -\text{ArcCos}\left[\frac{648000\ \left(-194400\ \pi + \pi^3 + 34992000\ \text{Sin}\left[\frac{\pi}{180}\right]\right)}{\pi^5}\right]\right\},\right.$
$\left.\left\{z \to \text{ArcCos}\left[\frac{648000\ \left(-194400\ \pi + \pi^3 + 34992000\ \text{Sin}\left[\frac{\pi}{180}\right]\right)}{\pi^5}\right]\right\}\right\}$

and then approximate the result with `N`.

```
In[350]:= N[cz]
Out[350]= {{z → -0.00384232},
           {z → 0.00384232}}
```

□

2.4.7 Other Series

In calculus, we learn that the power series $f(x) = \sum_{k=0}^{\infty} a_k \left(x - x_0\right)^k$ is differentiable and integrable on its interval of convergence. However, for series that are not power series this result is not generally true. For example, in more advanced courses, we learn that the function

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sin\left(3^k x\right)$$

is continuous for all values of x but nowhere differentiable. We can use Mathematica to help us see why this function is not differentiable. Let

$$f_n(x) = \sum_{k=0}^{n} \frac{1}{2^k} \sin\left(3^k x\right).$$

Notice that $f_n(x)$ is defined recursively by $f_0(x) = \sin x$ and $f_n(x) = f_{n-1}(x) + \frac{1}{2^n} \sin(3^n x)$. We use Mathematica to recursively define $f_n(x)$.

```
In[351]:= f[n_] := f[n] = f[n - 1] + Sin[3^n x]/2^n;
          f[0] = Sin[x];
```

We define $f_n(x)$ using the form

```
f[n_]:=f[n]=...
```

so that Mathematica "remembers" the values it computes. Thus, to compute `f[5]`, Mathematica uses the previously computed values, namely `f[4]`, to compute `f[5]`. Note that we can produce the same results by defining $f_n(x)$ with the command

```
f[n_]:=...
```

However, the disadvantage of defining $f_n(x)$ in this manner is that Mathematica does not "remember" the previously computed values and thus takes longer to compute $f_n(x)$ for larger values of n.

Next, we use `Table` to generate $f_3(x)$, $f_6(x)$, $f_9(x)$, and $f_{12}(x)$.

```
In[352]:= posums = Table[f[n], {n, 3, 12, 3}]
```

```
Out[352]= {Sin[x] + 1/2 Sin[3 x] + 1/4 Sin[9 x] + 1/8 Sin[27 x],
   Sin[x] + 1/2 Sin[3 x] + 1/4 Sin[9 x] + 1/8 Sin[27 x] +
    1/16 Sin[81 x] + 1/32 Sin[243 x] + 1/64 Sin[729 x], Sin[x] +
    1/2 Sin[3 x] + 1/4 Sin[9 x] + 1/8 Sin[27 x] + 1/16 Sin[81 x] +
    1/32 Sin[243 x] + 1/64 Sin[729 x] + 1/128 Sin[2187 x] +
    1/256 Sin[6561 x] + 1/512 Sin[19683 x], Sin[x] + 1/2 Sin[3 x] +
    1/4 Sin[9 x] + 1/8 Sin[27 x] + 1/16 Sin[81 x] + 1/32 Sin[243 x] +
    1/64 Sin[729 x] + 1/128 Sin[2187 x] + 1/256 Sin[6561 x] +
    1/512 Sin[19683 x] + Sin[59049 x]/1024 + Sin[177147 x]/2048 +
    Sin[531441 x]/4096}
```

We now graph each of these functions and show the results as a graphics array with `GraphicsArray` in Figure 2.35.

```
In[353]:= somegraphs = Map[Plot[#, {x, 0, 3π},
                 DisplayFunction-> Identity]&, posums];
          toshow = Partition[somegraphs, 2];
          Show[GraphicsArray[toshow]]
```

From these graphs, we see that for large values of n, the graph of $f_n(x)$, although actually smooth, appears ”jagged” and thus we might suspect that $f(x) = \lim_{n\to\infty} f_n(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sin\left(3^k x\right)$ is indeed continuous everywhere but nowhere differentiable.

2.5 Multi-Variable Calculus

Mathematica is useful is investigating functions involving more than one variable. In particular, the graphical analysis of functions that depend on two (or more) variables is enhanced with the help of Mathematica's graphics capabilities.

2.5.1 Limits of Functions of Two Variables

Mathematica's graphics and numerical capabilities are helpful in investigating limits of functions of two variables.

Example 61. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist.

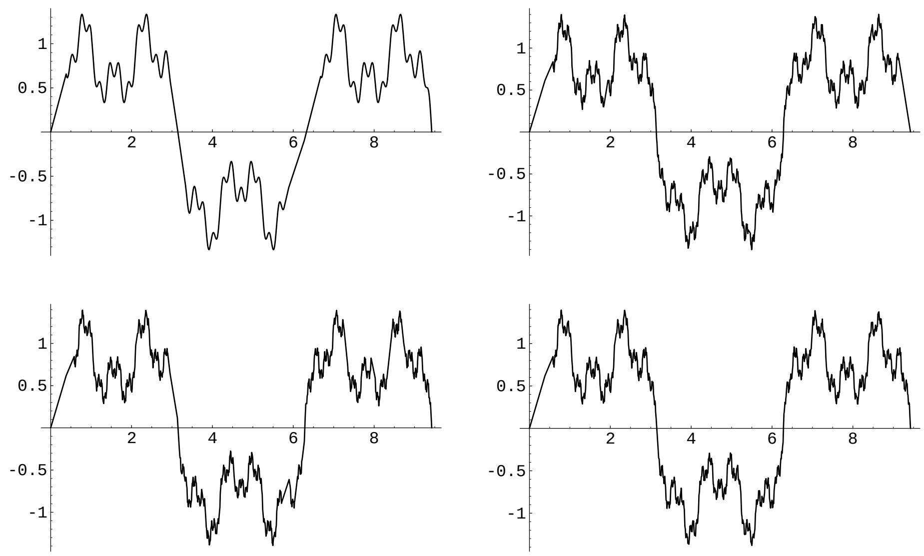


Figure 2.35: Approximating a function that is continuous everywhere but nowhere differentiable

Solution. We begin by defining $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$. Next, we use `Plot3D` to graph $z = f(x, y)$ for $-1/2 \le x \le 1/2$ and $-1/2 \le y \le 1/2$. `ContourPlot` is used to graph several level curves on the same rectangle. (See Figure 2.36.) (To define a function of two variables, $f(x, y) = \textit{expression in } x \textit{ and } y$, enter `f[x_,y_]=expression in x and y`. `Plot3D[f[x,y],{a,x,b},{y,c,d}]` generates a basic graph of $z = f(x, y)$ for $a \le x \le b$ and $c \le y \le d$)

```
In[354]:= f[x_, y_] = (x^2 - y^2)/(x^2 + y^2);
          p1 = Plot3D[f[x, y], {x, -0.5, 0.5}, {y, -0.5, 0.5},
              PlotPoints- > {40, 40}, DisplayFunction- > Identity];
          p2 = ContourPlot[f[x, y],
              {x, -0.5, 0.5}, {y, -0.5, 0.5}, PlotPoints- > 40,
              ContourShading- > False, Axes- > Automatic,
              AxesOrigin- > {0, 0}, DisplayFunction- > Identity];
          Show[GraphicsArray[{p1, p2}]]
```

From the graph of the level curves, we suspect that the limit does not exist because we see that near $(0, 0)$, $z = f(x, y)$ attains many different values. We obtain further evidence that the limit does not exist by computing the value of $z = f(x, y)$ for various points chosen randomly near $(0, 0)$. We use `Table` and `Random` to generate 13 ordered triples $(x, y, f(x, y))$ for x and y "close to" 0. Because `Random` is included in the calculation, your results will almost certainly be different from those here. The first column corresponds to the x-coordinate, the second column the y- coordinate, and the third column the value of $z = f(x, y)$.

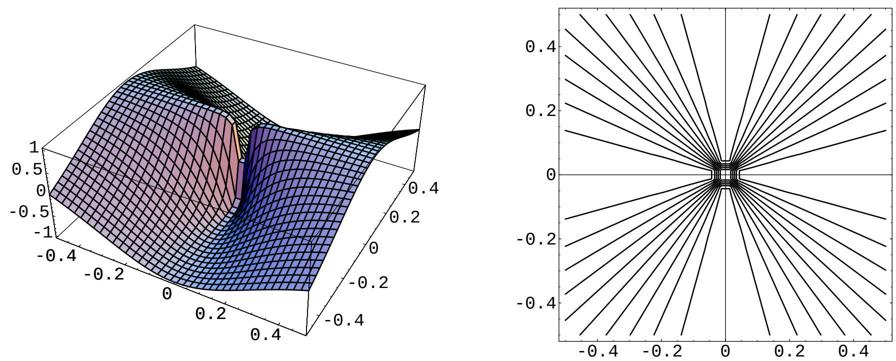


Figure 2.36: (a) 3-dimensional and (b) contour plots of $f(x, y)$

```
In[355]:= r[n_] := {Random[Real, {-10^(-n), 10^(-n)}],
              Random[Real, {-10^(-n), 10^(-n)}]}

In[356]:= r[1]

Out[356]= {5.25152, 9.37514}

In[357]:= toevaluate = Table[r[n], {n, 1, 15}]

Out[357]= {{0.043922, 0.0768676},
            {-0.00775639, 0.0039307},
            {-0.0000561454,
             -0.0000790007},
            {0.0000536954, 0.0000373069},
            {3.24752 10^-6, 7.41243 10^-6},
            {1.70105 10^-7, -6.6412 10^-7},
            {-1.15231 10^-9, -8.69882 10^-8},
            {3.85914 10^-9, 4.18814 10^-9},
            {-4.07047 10^-10,
             7.06248 10^-10},
            {8.19068 10^-11, 4.65551 10^-11},
            {1.67581 10^-12,
             -8.23982 10^-12},
            {8.47593 10^-13,
             -8.23785 10^-13},
            {7.28361 10^-14,
             -5.92658 10^-14},
            {6.23232 10^-15,
             -2.16855 10^-15},
            {-2.15493 10^-16,
             4.86343 10^-16}}

In[358]:= Map[f[#[[1]], #[[2]]]&, toevaluate]
```

```
Out[358]= {-0.507731,
           0.591324, -0.328828,
           0.348863, -0.677926,
           -0.876866, -0.999649,
           -0.0816327, -0.501298,
           0.511638, -0.920559,
           0.0284831, 0.203308,
           0.784009, -0.671783}
```

From the third column, we see that $z = f(x, y)$ does not appear to approach any particular value for points chosen randomly near $(0, 0)$. In fact, along the line $y = mx$ we see that

We choose lines of the form $y = mx$ because near $(0, 0)$ the level curves of $z = f(x, y)$ look like lines of the form $y = mx$.

$$f(x, y) = f(x, mx) = \frac{1 - m^2}{1 + m^2}.$$

Hence as $(x, y) \to (0, 0)$ along $y = mx$, $f(x, y) = f(x, mx) \to \frac{1-m^2}{1+m^2}$. Thus, $f(x, y)$ does not have a limit as $(x, y) \to (0, 0)$.

```
In[359]:= v1 = Simplify[f[x, m x]]

          v1 /.m- > 0

          v1/.m- > 1

          v1 /. m- > 1/2
```

Out[359]= $\frac{1 - m^2}{1 + m^2}$

Out[359]= 1

Out[359]= 0

Out[359]= $\frac{3}{5}$

□

In some cases, you can establish that a limit does not exist by converting to polar coordinates. For example, in polar coordinates, $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$ becomes $f(r\cos\theta, r\sin\theta) = 2\cos^2\theta - 1$

```
In[360]:= Simplify[f[r Cos[t], r Sin[t]]]
Out[360]= Cos[2 t]
```

and

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0} f(r\cos\theta, r\sin\theta) = \lim_{r\to 0} 2\cos^2\theta - 1 = 2\cos^2\theta - 1 = \cos 2\theta$$

depends on θ.

2.5.2 Partial and Directional Derivatives

Partial derivatives of functions of two or more variables are computed with Mathematica using `D`. For $z = f(x, y)$,

1. `D[f[x,y],x]` computes $\frac{\partial f}{\partial x} = f_x(x, y)$,
2. `D[f[x,y],y]` computes $\frac{\partial f}{\partial y} = f_x(x, y)$,
3. `D[f[x,y],{x,n}]` computes $\frac{\partial^n f}{\partial x^n}$, and
4. `D[f[x,y],y,x]` computes $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}(x, y)$, and
5. `D[f[x,y],{x,n},{y,m}]` computes $\frac{\partial^{n+m} f}{\partial^n x \partial^m y}$.

The calculations are carried out similarly for functions of more than two variables.

Example 62. Calculate $f_x(x, y)$, $f_y(x, y)$, $f_{xy}(x, y)$, $f_{yx}(x, y)$, $f_{xx}(x, y)$, and $f_{yy}(x, y)$ if $f(x, y) = \sin \sqrt{x^2 + y^2 + 1}$.

Solution. After defining $f(x, y) = \sin \sqrt{x^2 + y^2 + 1}$,

```
In[361]:= f[x_, y_] = Sin[Sqrt[x^2 + y^2 + 1]];
```

we illustrate the use of `D` to compute the partial derivatives. Entering

```
In[362]:= D[f[x, y], x]
```

$$\text{Out[362]=} \frac{x \operatorname{Cos}\left[\sqrt{1 + x^2 + y^2}\right]}{\sqrt{1 + x^2 + y^2}}$$

computes $f_x(x, y)$.

Entering

```
In[363]:= D[f[x, y], y]
```

$$\text{Out[363]=} \frac{y \operatorname{Cos}\left[\sqrt{1 + x^2 + y^2}\right]}{\sqrt{1 + x^2 + y^2}}$$

computes $f_y(x, y)$.

Entering

```
In[364]:= D[f[x, y], x, y]//Together
```

$$\text{Out[364]=} \frac{-x\, y \operatorname{Cos}\left[\sqrt{1 + x^2 + y^2}\right] - x\, y \sqrt{1 + x^2 + y^2} \operatorname{Sin}\left[\sqrt{1 + x^2 + y^2}\right]}{(1 + x^2 + y^2)^{3/2}}$$

computes $f_{yx}(x, y)$.

Entering

```
In[365]:= D[f[x, y], y, x]//Together
```

```
Out[365]= (-x y Cos[Sqrt[1 + x^2 + y^2]] - x y Sqrt[1 + x^2 + y^2] Sin[Sqrt[1 + x^2 + y^2]]) / (1 + x^2 + y^2)^(3/2)
```

computes $f_{xy}(x,y)$. Remember that under appropriate assumptions, $f_{xy}(x,y) = f_{yx}(x,y)$.

Entering

```
In[366]:= D[f[x, y], {x, 2}]//Together
Out[366]= 1/(1 + x^2 + y^2)^(3/2) (Cos[Sqrt[1 + x^2 + y^2]] + y^2 Cos[Sqrt[1 + x^2 + y^2]] -
            x^2 Sqrt[1 + x^2 + y^2] Sin[Sqrt[1 + x^2 + y^2]])
```

computes $f_{xx}(x,y)$.

Entering

```
In[367]:= D[f[x, y], {y, 2}]//Together
Out[367]= 1/(1 + x^2 + y^2)^(3/2) (Cos[Sqrt[1 + x^2 + y^2]] + x^2 Cos[Sqrt[1 + x^2 + y^2]] -
            y^2 Sqrt[1 + x^2 + y^2] Sin[Sqrt[1 + x^2 + y^2]])
```

computes $f_{yy}(x,y)$. □

The **directional derivative** of $z = f(x,y)$ in the direction of the unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is

The vectors $\mathbf{i}$ and $\mathbf{j}$ are defined by $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$.

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta,$$

provided that $f_x(x,y)$ and $f_y(x,y)$ both exist.

If $f_x(x,y)$ and $f_y(x,y)$ both exist, the **gradient** of $f(x,y)$ is the vector-valued function

Calculus of vector-valued functions is discussed in more detail in the next section.

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = \left\langle f_x(x,y), f_y(x,y)\right\rangle.$$

Notice that if $\mathbf{u} = \langle\cos\theta, \sin\theta\rangle$,

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \langle\cos\theta, \sin\theta\rangle.$$

Example 63. Let

$$f(x,y) = 6x^2y - 3x^4 - 2y^3.$$

(a) Find $D_{\mathbf{u}} f(x,y)$ in the direction of $\mathbf{v} = \langle 3, 4\rangle$. (b) Compute

$$D_{\langle 3/4, 4/5\rangle} f\left(\frac{1}{3}\sqrt{9+3\sqrt{3}}, 1\right).$$

(c) Find an equation of the line tangent to the graph of $6x^2y - 3x^4 - 2y^3 = 0$ at the point $\left(\frac{1}{3}\sqrt{9+3\sqrt{3}}, 1\right)$.

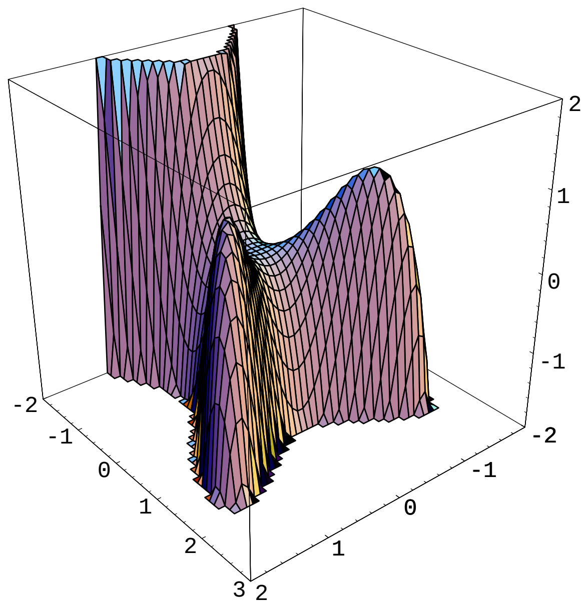


Figure 2.37: $f(x, y) = 6x^2y - 3x^4 - 2y^3$ for $-2 \le x \le 2$ and $-2 \le y \le 3$

Solution. After defining $f(x, y) = 6x^2y-3x^4-2y^3$, we graph $z = f(x, y)$ with `Plot3D` in Figure 2.37, illustrating the `PlotPoints`, `PlotRange`, and `ViewPoint` options.

```
In[368]:= f[x_, y_] = 6x^2y - 3x^4 - 2y^3;
          Plot3D[f[x, y], {x, -2, 2},
            {y, -2, 3}, PlotPoints- > 50, PlotRange- >
              {{-2, 2}, {-2, 3}, {-2, 2}}, BoxRatios- > {1, 1, 1},
            ViewPoint- > {1.887,  2.309,  1.6}, ClipFill- > None]
```

(a) A unit vector, **u**, in the same direction as **v** is

$$\mathbf{v} = \left\langle \frac{3}{\sqrt{3^2+4^2}}, \frac{4}{\sqrt{3^2+4^2}} \right\rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

```
In[369]:= v = {3, 4};
          u = v/Sqrt[v.v]
```

Out[369]= $\left\{\frac{3}{5}, \frac{4}{5}\right\}$

Then, $D_{\mathbf{u}} f(x, y) = \left\langle f_x(x, y), f_y(x, y) \right\rangle \cot \mathbf{u}$, caculated in `du`.

```
In[370]:= gradf = {D[f[x, y], x], D[f[x, y], y]}
```

Out[370]= $\{-12\ x^3 + 12\ x\ y, 6\ x^2 - 6\ y^2\}$

```
In[371]:= du = Simplify[grad.u]
```

Out[371]= $-\frac{12}{5}\ (-2\ x^2 + 3\ x^3 - 3\ x\ y + 2\ y^2)$

(b) $D_{\langle 3/4,4/5\rangle} f\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)$ is calculated by evaluating du if $x = \frac{1}{3}\sqrt{9+3\sqrt{3}}$ and $y = 1$.

```
In[372]:= du1 = du/.{x- > 1/3Sqrt[9 + 3Sqrt[3]], y- > 1}//Simplify
```

$$\text{Out[372]=}\ -\frac{4}{5}\sqrt{3}\left(-2+\sqrt{3\left(3+\sqrt{3}\right)}\right)$$

(c) The gradient is evaluated if $x = \frac{1}{3}\sqrt{9+3\sqrt{3}}$ and $y = 1$.

```
In[373]:= nvec =
            gradf/.{x- > 1/3Sqrt[9 + 3Sqrt[3]], y- > 1}//Simplify
```

$$\text{Out[373]=}\ \left\{-4\sqrt{3+\sqrt{3}},\ 2\sqrt{3}\right\}$$

Generally, $\nabla f(x,y)$ is perpendicular to the level curves of $z = f(x,y)$, so

$$\texttt{nvec} = \nabla f\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right) = \left\langle f_x\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right), f_y\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)\right\rangle$$

is perpendicular to $f(x,y) = 0$ at the point $\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)$. Thus, an equation of the line tangent to the graph of $f(x,y) = 0$ at the point $\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)$ is

An equation of the line L containing (x_0, y_0) and perpendicular to $\mathbf{n} = \langle a,b\rangle$ is $a(x-x_0)+b(y-y_0) = 0$.

$$f_x\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)\left(x-\frac{1}{3}\sqrt{9+3\sqrt{3}}\right) + f_y\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)(y-1) = 0,$$

which we solve for y with Solve. We confirm this result by graphing $f(x,y) = 0$ using ContourPlot with the Contours->{0} option in conf and then graphing the tangent line in tanplot. tanplot and conf are shown together with Show in Figure 2.38.

```
In[374]:= conf = ContourPlot[
              f[x, y], {x, -2, 2}, {y, -2, 2}, Contours- > {0},
              PlotPoints- > 60, ContourShading- > False,
              Frame- > False, Axes- > Automatic, AxesOrigin- > {0, 0},
              DisplayFunction- > Identity];

In[375]:= tanline = Solve[nvec[[1]]
                 (x - 1/3Sqrt[9 + 3Sqrt[3]]) + nvec[[2]](y - 1) == 0,
              y]
```

$$\text{Out[375]=}\ \left\{\left\{y \to -\frac{-2\sqrt{3}-4\sqrt{3+\sqrt{3}}\left(-\frac{1}{3}\sqrt{9+3\sqrt{3}}+x\right)}{2\sqrt{3}}\right\}\right\}$$

```
In[376]:= tanplot = Plot[Evaluate[y/.tanline],
              {x, -2, 2}, DisplayFunction- > Identity];
          Show[conf, tanplot, DisplayFunction- > $DisplayFunction,
            PlotRange- > {{-2, 2}, {-2, 3}}, AspectRatio- > Automatic]
```

□

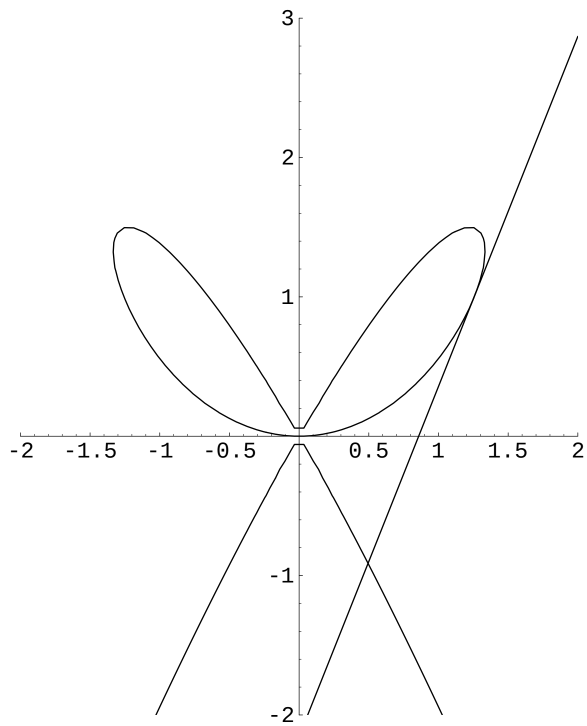


Figure 2.38: Level curves of $f(x, y)$

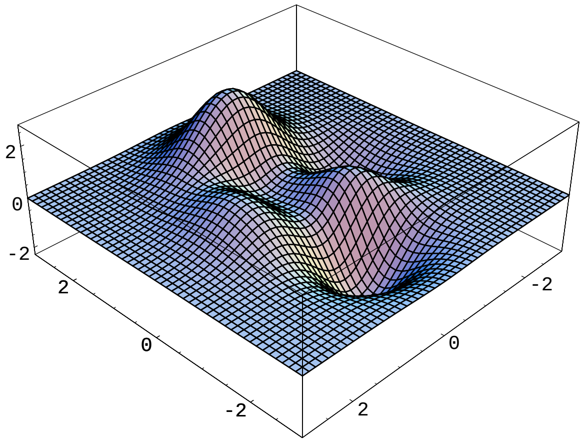


Figure 2.39: $f(x,y)$ for $-3 \le x \le 3$ and $-3 \le y \le 2$

Example 64. Let

$$f(x,y) = (y-1)^2 e^{-(x+1)^2-y^2} - \frac{10}{3}\left(-x^5 + \frac{1}{5}y - y^3\right)e^{-x^2-y^2} - \frac{1}{9}e^{-x^2-(y+1)^2}.$$

Calculate $\nabla f(x,y)$ and then graph $\nabla f(x,y)$ together with several level curves of $f(x,y)$.

Solution. We begin by defining and graphing $z = f(x,y)$ with `Plot3D` in Figure 2.39.

```
In[377]:= f[x_, y_] = (y - 1)^2 Exp[-(x + 1)^2 - y^2] -
              10/3 (-x^5 + 1/5 y - y^3) Exp[-x^2 - y^2] -
              1/9 Exp[-x^2 - (y + 1)^2];

In[378]:= Plot3D[f[x, y], {x, -3, 3}, {y, -3, 3}, PlotPoints-> 50,
            ViewPoint-> {-1.99, 2.033, 1.833}, PlotRange-> All];
          conf = ContourPlot[f[x, y], {x, -3, 3},
              {y, -3, 3}, PlotPoints-> 60, ContourShading-> False,
              Frame-> False, Axes-> Automatic, AxesOrigin-> {0, 0},
              DisplayFunction-> Identity];
```

In the three-dimensional plot, notice that z appears to have five relative extrema: three relative maxima and three relative minima. We also graph several level curves of $f(x,y)$ with `ContourPlot` and name the resulting graphic `conf`. The graphic is not displayed because we include the option `DisplayFunction->Identity`.

Next we calculate $f_x(x,y)$ and $f_y(x,y)$ using `Simplify` and `D`. The gradient is the vector-valued function $\langle f_x(x,y), f_y(x,y)\rangle$.

```
In[379]:= gradf = {D[f[x, y], x], D[f[x, y], y]}//Simplify
```

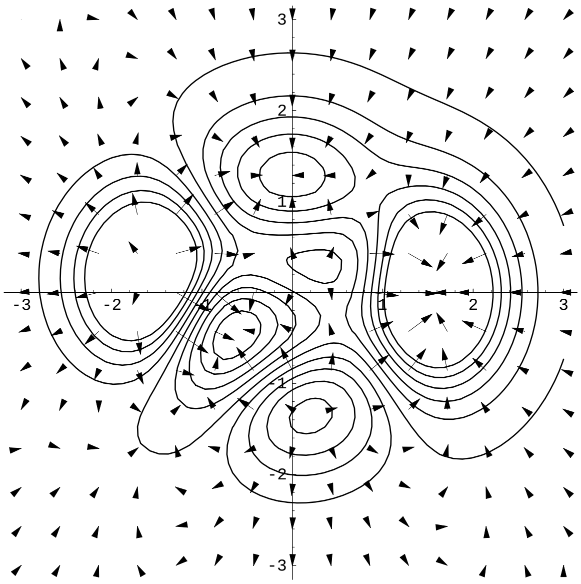


Figure 2.40: Contour plot of $f(x,y)$ along with several gradient vectors

Out[379]=
$$\left\{\frac{2}{9}\left(e^{-x^2-(1+y)^2}\ x+75\ e^{-x^2-y^2}\ x^4-9\ e^{-(1+x)^2-y^2}\ (1+x)\ (-1+y)^2-6\ e^{-x^2-y^2}\ x\ \left(5\ x^5-y+5\ y^3\right)\right),\right.$$
$$\left.-\frac{2}{9}\ e^{1+x^2+y^2-2\ (1+x+x^2+y+y^2)}\left(-e^{2\ x}+9\ e^{2\ y}+3\ e^{1+2\ x+2\ y}+e^{2\ x}\left(-1+30\ e^{1+2\ y}\ x^5\right)\ y-3\ e^{2\ y}\left(6+17\ e^{1+2\ x}\right)\ y^2+9\ e^{2\ y}\ y^3+30\ e^{1+2\ x+2\ y}\ y^4\right)\right\}$$

To graph the gradient, we use `PlotGradientField`, which is contained in the `PlotField` package. We use `PlotGradientField` to graph the gradient naming the resulting graphic `gradf`. `gradf` and `conf` are displayed together using `Show`.

In[380]:=
```
<< Graphics`PlotField`

gradfplot = PlotGradientField[f[x, y],
    x, -3, 3}, {y, -3, 3}, DisplayFunction- > Identity];
Show[c
  onf, gradfplot, DisplayFunction- > $DisplayFunction]
```

In the result (see Figure 2.40), notice that the gradient is perpendicular to the level curves; the gradient is pointing in the direction of maximal increase of $z = f(x,y)$. □

Classifying Critical Points

Let $z = f(x,y)$ be a real-valued function of two variables with continuous second-order partial derivatives. A **critical point** of $z = f(x,y)$ is a point (x_0, y_0) in the interior of the domain of $z = f(x, y)$ for which

$$f_x\left(x_0, y_0\right) = 0 \qquad \text{and} \qquad f_y\left(x_0, y_0\right) = 0.$$

Critical points are classified by the *Second Derivatives* (or *Partials*) *test*.

Theorem 16 (Second Derivatives Test). *Let $\left(x_0, y_0\right)$ be a critical point of a function $z = f(x, y)$ of two variables and let*

$$d = f_{xx}\left(x_0, y_0\right)\, f_{yy}\left(x_0, y_0\right) - \left[f_{xy}\left(x_0, y_0\right)\right]^2. \tag{2.26}$$

1. *If $d > 0$ and $f_{xx}\left(x_0, y_0\right) > 0$, then $z = f(x, y)$ has a **relative** (or **local**) **minimum** at $\left(x_0, y_0\right)$.*
2. *If $d > 0$ and $f_{xx}\left(x_0, y_0\right) < 0$, then $z = f(x, y)$ has a **relative** (or **local**) **maximum** at $\left(x_0, y_0\right)$.*
3. *If $d < 0$, then $z = f(x, y)$ has a **saddle point** at $\left(x_0, y_0\right)$.*
4. *If $d = 0$, no conclusion can be drawn and $\left(x_0, y_0\right)$ is called a **degenerate critical point**.*

Example 65. Find the relative maximum, relative minimum, and saddle points of $f(x, y) = -2x^2 + x^4 + 3y - y^3$.

Solution. After defining $f(x, y)$, the critical points are found with `Solve` and named `critpts`.

```
In[381]:= f[x_, y_] = -2x^2 + x^4 + 3y - y^3;
          critpts =
           Solve[{D[f[x, y], x] == 0, D[f[x, y], y] == 0}, {x, y}]
Out[381]= {{x → -1, y → -1}, {x → -1, y → 1}, {x → 0, y → -1},
           {x → 0, y → 1}, {x → 1, y → -1}, {x → 1, y → 1}}
```

We then define `dfxx`. Given $\left(x_0, y_0\right)$, `dfxx` $\left(x_0, y_0\right)$ returns the ordered quadruple x_0, y_0, (2.26) evaluated at $\left(x_0, y_0\right)$, and $f_{xx}\left(x_0, y_0\right)$.

```
In[382]:= dfxx[x0_, y0_] =
           {x0, y0, D[f[x, y], {x, 2}]D[f[x, y], {y, 2}]-
                D[f[x, y], x, y]^2/.{x- > x0, y- > y0},
            D[f[x, y], {x, 2}]/.{x- > x0, y- > y0}}
Out[382]= {x0, y0, -6 (-4 + 12 x0^2) y0, -4 + 12 x0^2}
```

For example,

```
In[383]:= dfxx[0, 1]
Out[383]= {0, 1, 24, -4}
```

shows us that a relative maximum occurs at $(0, 1)$. We then use `/.` (`ReplaceAll`) to substitute the values in each element of `critpts` into `dfxx`.

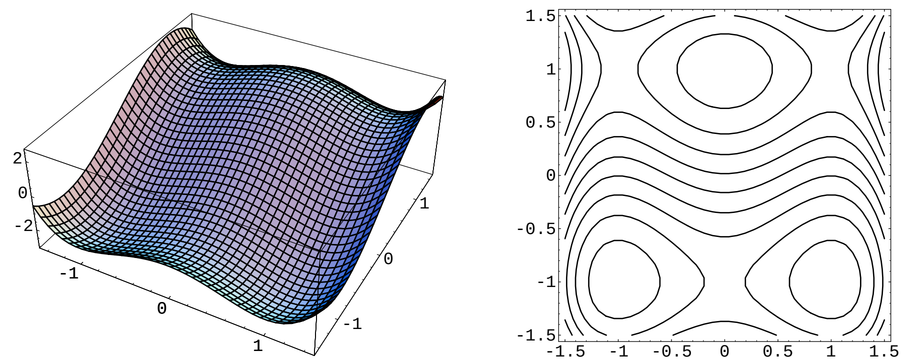


Figure 2.41: (a) Three-dimensional and (b) contour plots of $f(x, y)$

```
In[384]:= dfxx[x, y]/.critpts
Out[384]= {{-1, -1, 48, 8}, {-1, 1, -48, 8}, {0, -1, -24, -4},
            {0, 1, 24, -4}, {1, -1, 48, 8}, {1, 1, -48, 8}}
```

From the result, we see that $(0, 1)$ results in a relative maximum, $(0, -1)$ results in a saddle, $(1, 1)$ results in a saddle, $(1, -1)$ results in a relative minimum, $(-1, 1)$ results in a saddle, and $(1, -1)$ results in a relative minimum. We confirm these results graphically with a three-dimensional plot generated with `Plot3D` and a contour plot generated with `ContourPlot` in Figure 2.41.

```
In[385]:= p1 = Plot3D[f[x, y], {x, -3/2, 3/2}, {y, -3/2, 3/2},
              PlotPoints- > 40, DisplayFunction- > Identity];
          p2 = ContourPlot[f[x, y],
              {x, -3/2, 3/2}, {y, -3/2, 3/2}, PlotPoints- > 40,
              ContourShading- > False, DisplayFunction- > Identity];
          Show[GraphicsArray[{p1, p2}]]
```

In the contour plot, notice that near relative extrema, the level curves look like circles while near saddles they look like hyperbolas. □

If the Second Derivatives test fails, graphical analysis is especially useful.

Example 66. Find the relative maximum, relative minimum, and saddle points of $f(x, y) = x^2 + x^2y^2 + y^4$.

Solution. Initially we proceed in the exact same manner as in the previous example: we define $f(x, y)$ and compute the critical points. Several complex solutions are returned, which we ignore.

```
In[386]:= f[x_, y_] = x^2 + x^2y^2 + y^4;
          critpts =
            Solve[{D[f[x, y], x] == 0, D[f[x, y], y] == 0}, {x, y}]
```

```
Out[386]= {{x → 0, y → 0}, {x → -√2, y → -i}, {x → -√2, y → i},
           {x → √2, y → -i}, {x → √2, y → i}, {y → 0, x → 0},
           {y → 0, x → 0}}
```

We then compute the value of (2.26) at the real critical point, and the value of $f_{xx}(x, y)$ at this critical point.

```
In[387]:= dfxx[x0_, y0_] =
            {x0, y0, D[f[x, y], {x, 2}]D[f[x, y], {y, 2}]-
                D[f[x, y], x, y]^2/.{x- > x0, y- > y0},
             D[f[x, y], {x, 2}]/.{x- > x0, y- > y0}}
Out[387]= {x0, y0, -16 x0^2 y0^2 + (2 + 2 y0^2) (2 x0^2 + 12 y0^2), 2 + 2 y0^2}

In[388]:= dfxx[0, 0]
Out[388]= {0, 0, 0, 2}
```

The result shows us that the Second Derivatives test fails at $(0, 0)$.

```
In[389]:= p1 = Plot3D[f[x, y], {x, -1, 1}, {y, -1, 1}, PlotPoints- > 40,
              DisplayFunction- > Identity, BoxRatios- > Automatic];
          p2 = ContourPlot[f[x, y], {x, -1, 1},
              {y, -1, 1}, PlotPoints- > 40, Contours- > 20,
              ContourShading- > False, DisplayFunction- > Identity];
          Show[GraphicsArray[{p1, p2}]]
```

However, the contour plot of $f(x, y)$ near $(0, 0)$ indicates that an extreme value occurs at $(0, 0)$. The three-dimensional plot shows that $(0, 0)$ is a relative minimum. (See Figure 2.42.) □

Tangent Planes

Let $z = f(x, y)$ be a real-valued function of two variables. If both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, then an equation of the plane tangent to the graph of $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0, \tag{2.27}$$

where $z_0 = f(x_0, y_0)$. Solving for z yields the function (of two variables)

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0. \tag{2.28}$$

Symmetric equations of the line perpendicular to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) are given by

$$\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - z_0}{-1} \tag{2.29}$$

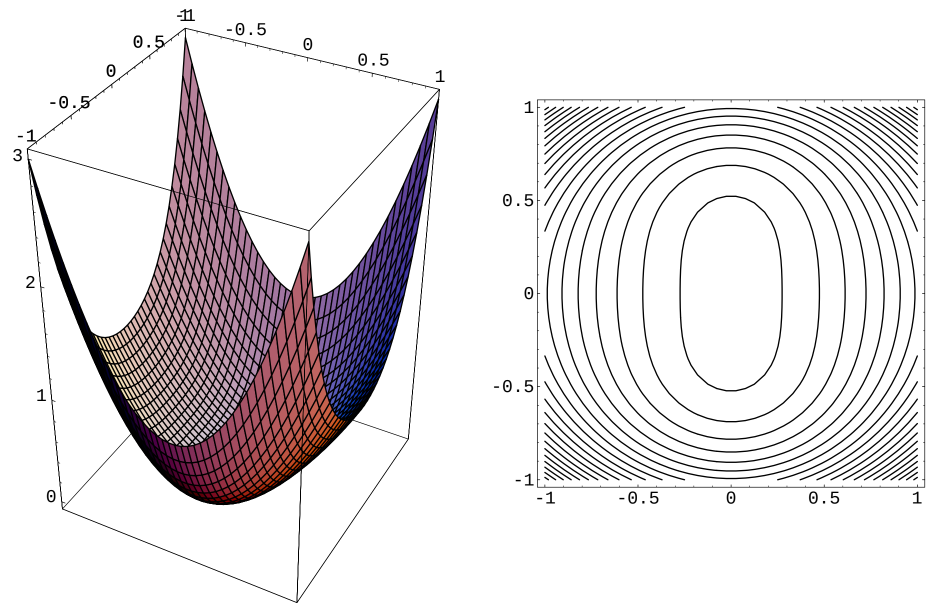


Figure 2.42: (a) Three-dimensional and (b) contour plots of $f(x, y)$

and parametric equations are

$$\begin{cases} x = x_0 + f_x(x_0, y_0)\,t \\ y = y_0 + f_y(x_0, y_0)\,t \\ z = z_0 - t \end{cases} \tag{2.30}$$

The plane tangent to the graph of $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is the "best" linear approximation of $z = f(x, y)$ near $(x, y) = (x_0, y_0)$ in the same way as the line tangent to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ is the "best" linear approximation of $y = f(x)$ near $x = x_0$.

Example 67. Find an equation of the plane tangent and normal line to the graph of at $f(x, y) = 4 - \frac{1}{4}(2x^2 + y^2)$ at the point $(1, 2, 5/2)$.

Solution. We define $f(x, y)$ and compute $f_x(1, 2)$ and $f_y(1, 2)$.

```
In[390]:= f[x_, y_] = 4 - 1/4(2x^2 + y^2);
          f[1, 2]

          dx = D[f[x, y], x]/.{x- > 1, y- > 2}

          dy = D[f[x, y], y]/.{x- > 1, y- > 2}
```

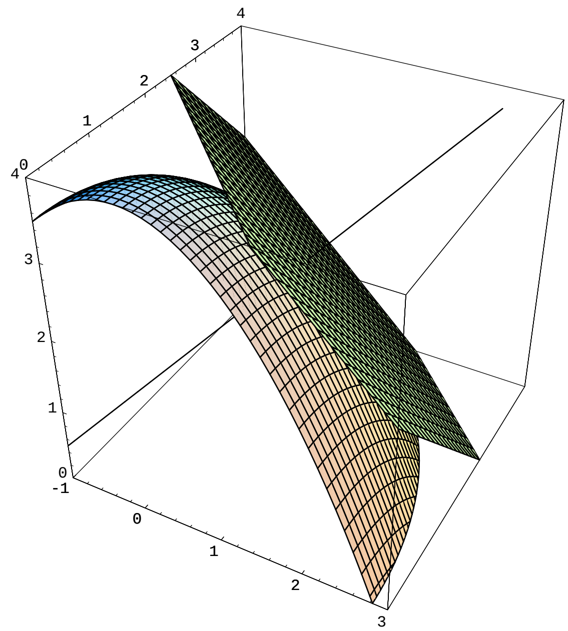


Figure 2.43: Graph of $f(x, y)$ with a tangent plane and normal line

```
Out[390]= 5/2
Out[390]= -1
Out[390]= -1
```

Using (2.28), an equation of the tangent plane is $z = -1(x-1)-1(y-2)+f(1,2)$. Using (2.30), parametric equations of the normal line are $x = 1-t$, $y = 2-t$, $z = f(1,2)-t$. We confirm the result graphically by graphing $f(x, y)$ together with the tangent plane in `p1` using `Plot3D`. We use `ParametricPlot3D` to graph the normal line in `p2` and then display `p1` and `p2` together with `Show` in Figure 2.43.

```
In[391]:= p1 = Plot3D[f[x, y], {x, -1, 3}, {y, 0, 4},
              DisplayFunction- > Identity, PlotPoints- > 40];
          p2 = Plot3D[dx (x - 1) + dy (y - 2) + f[1, 2], {x, -1, 3},
              {y, 0, 4}, DisplayFunction- > Identity, PlotPoints- > 30];
          p3 = ParametricPlot3D[{1 + dx t, 2 + dy t, f[1, 2] - t},
              {t, -4, 4}, DisplayFunction- > Identity];
          Show[p1, p2, p3, PlotRange- > {{-1, 3}, {0, 4}, {0, 4}},
            BoxRatios- > Automatic,
            DisplayFunction- > $DisplayFunction]
```

Because $z = -1(x-1)-1(y-2)+f(1,2)$ is the "best" linear approximation of $f(x, y)$ near $(1, 2)$, the graphs are very similar near $(1, 2)$ as shown in the three-dimensional plot. We also expect the level curves of each near $(1, 2)$ to be similar, which is confirmed with `ContourPlot` in Figure 2.44.

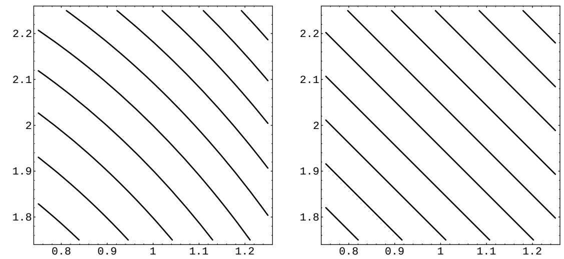


Figure 2.44: Zooming in near (1, 2)

```
In[392]:= p4 = ContourPlot[f[x, y], {x, 0.75, 1.25}, {y, 1.75, 2.25},
              ContourShading- > False, DisplayFunction- > Identity];
          p5 = ContourPlot[dx (x - 1) + dy (y - 2) + f[1, 2],
              {x, 0.75, 1.25}, {y, 1.75, 2.25},
              ContourShading- > False, DisplayFunction- > Identity];
          Show[GraphicsArray[{p4, p5}]]
```

□

Lagrange Multipliers

Certain types of optimization problems can be solved using the method of *Lagrange multipliers* that is based on the following theorem.

Theorem 17 (Lagrange's Theorem). *Let $z = f(x,y)$ and $z = g(x,y)$ be real-valued functions with continuous partial derivatives and let $z = f(x,y)$ have an extreme value at a point $\left(x_0, y_0\right)$ on the smooth constraint curve $g(x,y) = 0$. If $\nabla g\left(x_0, y_0\right) \neq \mathbf{0}$, then there is a real number λ satisfying*

$$\nabla f\left(x_0, y_0\right) = \lambda \nabla g\left(x_0, y_0\right) \tag{2.31}$$

Graphically, the points $\left(x_0, y_0\right)$ at which the extreme values occur correspond to the points where the level curves of $z = f(x,y)$ are tangent to the graph of $g(x,y) = 0$.

Example 68. Find the maximum and minimum values of $f(x,y) = xy$ subject to the constraint $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$.

Solution. For this problem, $f(x,y) = xy$ and $g(x,y) = \frac{1}{4}x^2 + \frac{1}{9}y^2 - 1$. Observe that parametric equations for $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$ are $x = 2\cos t$, $y = 3\sin t$, $0 \leq t \leq 2\pi$. In Figure 2.45, we use `ParametricPlot3D` to parametrically graph $g(x,y) = 0$ and $f(x,y)$ for x and y-values on the curve $g(x,y) = 0$ by graphing

$$\begin{cases} x = 2\cos t \\ y = 3\sin t \\ z = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} x = 2\cos t \\ y = 3\sin t \\ z = x \cdot y = 6\cos t \sin t \end{cases}$$

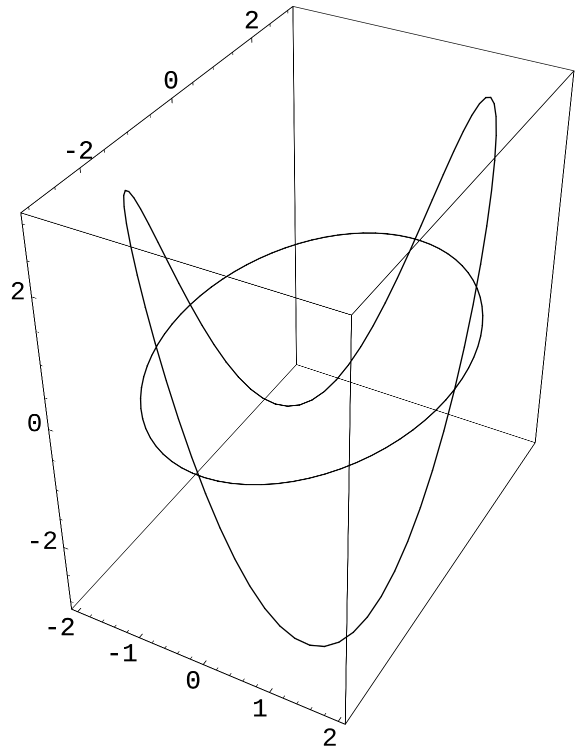


Figure 2.45: $f(x, y)$ on $g(x, y) = 0$

for $0 \le t \le 2\pi$. Our goal is to find the minimum and maximum values in Figure 2.45 and the points at which they occur.

```
In[393]:= f[x_, y_] = x y;
          g[x_, y_] = x^2/4 + y^2/9 - 1;
```

```
In[394]:= s1 = ParametricPlot3D[{2 Cos[t], 3 Sin[t], 0},
              {t, 0, 2π}, DisplayFunction- > Identity];
          s2 = ParametricPlot3D[{2 Cos[t], 3 Sin[t], 6 Cos[t] Sin[t]},
              {t, 0, 2π}, DisplayFunction- > Identity];
          Show[s1, s2, BoxRatios- > Automatic,
            DisplayFunction- > $DisplayFunction]
```

To implement the method of Lagrange multipliers, we compute $f_x(x, y)$, $f_y(x, y)$, $g_x(x, y)$, and $g_y(x, y)$ with `D`.

```
In[395]:= fx = D[f[x, y], x]

          fy = D[f[x, y], y]

          gx = D[g[x, y], x]

          gy = D[g[x, y], y]
```

Out[395]= y

Out[395]= x

Out[395]= $\frac{x}{2}$

Out[395]= $\frac{2\,y}{9}$

`Solve` is used to solve the system of equations (2.31):

$$f_x(x,y) = \lambda g_x(x,y)$$
$$f_y(x,y = \lambda g_y(x,y)$$
$$g(x,y) = 0$$

for x, y, and λ.

```
In[396]:= vals = Solve[{fx == λ gx, fy == λ gy, g[x, y] == 0},
            {x, y, λ}]
```

Out[396]= $\left\{\left\{\lambda \to -3, x \to -\sqrt{2}, y \to \frac{3}{\sqrt{2}}\right\}, \left\{\lambda \to -3, x \to \sqrt{2}, y \to -\frac{3}{\sqrt{2}}\right\}, \left\{\lambda \to 3, x \to -\sqrt{2}, y \to -\frac{3}{\sqrt{2}}\right\}, \left\{\lambda \to 3, x \to \sqrt{2}, y \to \frac{3}{\sqrt{2}}\right\}\right\}$

The corresponding values of $f(x,y)$ are found using `/.`.

```
In[397]:= n1 = {x, y, f[x, y]}/.vals
```

Out[397]= $\left\{\left\{-\sqrt{2}, \frac{3}{\sqrt{2}}, -3\right\}, \left\{\sqrt{2}, -\frac{3}{\sqrt{2}}, -3\right\}, \left\{-\sqrt{2}, -\frac{3}{\sqrt{2}}, 3\right\}, \left\{\sqrt{2}, \frac{3}{\sqrt{2}}, 3\right\}\right\}$

```
In[398]:= N[n1]
Out[398]= {{-1.41421, 2.12132, -3.},
           {1.41421, -2.12132, -3.},
           {-1.41421, -2.12132, 3.},
           {1.41421, 2.12132, 3.}}
```

We conclude that the maximum value $f(x,y)$ subject to the constraint $g(x,y) = 0$ is 3 and occurs at $\left(\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$ and $\left(-\sqrt{2}, -\frac{3}{2}\sqrt{2}\right)$. The minimum value is -3 and occurs at $\left(-\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$ and $\left(\sqrt{2}, -\frac{3}{2}\sqrt{2}\right)$. We graph several level curves of $f(x,y)$ and the graph of $g(x,y) = 0$ with `ContourPlot` and show the graphs together with `Show`. The minimum and maximum values of $f(x,y)$ subject to the constraint $g(x,y) = 0$ occur at

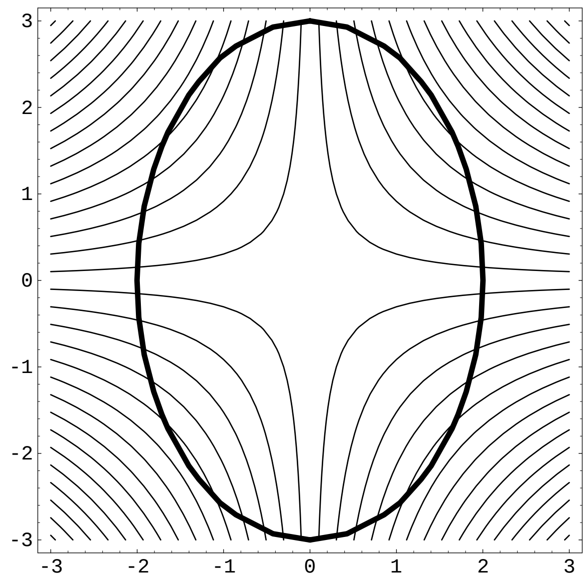


Figure 2.46: Level curves of $f(x,y)$ together with $g(x,y) = 0$

the points where the level curves of $f(x,y)$ are tangent to the graph of $g(x,y) = 0$ as illustrated in Figure 2.46.

```
In[399]:= cp1 = ContourPlot[f[x, y], {x, -3, 3},
              {y, -3, 3}, Contours- > 30, ContourShading- > False,
              PlotPoints- > 40, DisplayFunction- > Identity];
          cp2 = ContourPlot[
              g[x, y], {x, -3, 3}, {y, -3, 3}, Contours- > {0},
              ContourShading- > False, DisplayFunction- > Identity,
              ContourStyle- > Thickness[0.01]];
          Show[cp1, cp2, DisplayFunction- > $DisplayFunction]
```

□

2.5.3 Iterated Integrals

The `Integrate` command, used to compute single integrals, is used to compute iterated integrals. The command

`Integrate[f[x,y],{y,c,d},{x,a,b}]`

attempts to compute the iterated integral

$$\int_c^d \int_a^b f(x,y)\,dx\,dy. \qquad (2.32)$$

If Mathematica cannot compute the exact value of the integral, it is returned unevaluated, in which case numerical results may be more useful. The iterated integral (2.32) is numerically evaluated with the command `N` or

```
NIntegrate[f[x,y],{y,c,d},{x,a,b}]
```

Example 69. Evaluate each integral: (a) $\int_2^4 \int_1^2 \left(2xy^2 + 3x^2y\right)\, dx\, dy$ (b) $\int_0^2 \int_{y^2}^{2y} \left(3x^2 + y^3\right) dx\, dy$ (c) $\int_0^\infty \int_0^\infty xye^{-x^2-y^2}\, dy\, dx$ (d) $\int_0^\pi \int_0^\pi e^{\sin xy} dx\, dy$

Solution. (a) First we compute $\iint \left(2xy^2 + 3x^2y\right)\, dx\, dy$ with `Integrate`. Second, we compute $\int_2^4 \int_1^2 \left(2xy^2 + 3x^2y\right)\, dx\, dy$ with `Integrate`.

```
In[400]:= Integrate[2x y^2 + 3x^2 y, y, x]
```

$$\text{Out[400]= } \frac{x^3\ y^2}{2} + \frac{x^2\ y^3}{3}$$

```
In[401]:= Integrate[2x y^2 + 3x^2 y, {y, 2, 4}, {x, 1, 2}]
Out[401]= 98
```

(b) We illustrate the same commands as in (a), except we are integrating over a non-rectangular region.

```
In[402]:= Integrate[3x^2 + y^3, {x, y^2, 2y}]
```

$$\text{Out[402]= } 8\ y^3 + 2\ y^4 - y^5 - y^6$$

```
In[403]:= Integrate[3x^2 + y^3, y, {x, y^2, 2y}]
```

$$\text{Out[403]= } 2\ y^4 + \frac{2\ y^5}{5} - \frac{y^6}{6} - \frac{y^7}{7}$$

```
In[404]:= Integrate[3x^2 + y^3, {y, 0, 2}, {x, y^2, 2y}]
```

$$\text{Out[404]= } \frac{1664}{105}$$

(c) Improper integrals can be handled in the same way as proper integrals.

```
In[405]:= Integrate[x y Exp[-x^2 - y^2], x, y]
```

$$\text{Out[405]= } \frac{1}{4}\ e^{-x^2-y^2}$$

```
In[406]:= Integrate[x y Exp[-x^2 - y^2], {x, 0, ∞},
            {y, 0, ∞}]
```

$$\text{Out[406]= } \frac{1}{4}$$

(d) In this case, Mathematica cannot evaluate the integral exactly so we use `NIntegrate` to obtain an approximation.

```
In[407]:= Integrate[Exp[Sin[x y]], y, x]
```

$$\text{Out[407]= } \int\int e^{\text{Sin}[x\ y]}\, dx\, dy$$

```
In[408]:= NIntegrate[Exp[Sin[x y]], {y, 0, π}, {x, 0, π}]
Out[408]= 15.5092
```

□

Area, Volume, and Surface Area

Typical applications of iterated integrals include determining the area of a planar region, the volume of a region in three-dimensional space, or the surface area of a region in three-dimensional space. The area of the planar region R is given by

$$A = \iint_R dA. \tag{2.33}$$

If $z = f(x, y)$ has continuous partial derivatives on a closed region R, then the surface area of the portion of the surface that projects onto R is given by

$$SA = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA \tag{2.34}$$

If $f(x, y) \geq g(x, y)$ on R, the volume of the region between the graphs of $f(x, y)$ and $g(x, y)$ is

$$V = \iint_R (f(x, y) - g(x, y))\, dA \tag{2.35}$$

Example 70. Find the area of the region R bounded by the graphs of $y = 2x^2$ and $y = 1 + x^2$.

Solution. We begin by graphing $y = 2x^2$ and $y = 1 + x^2$ with `Plot` in Figure 2.47. The x-coordinates of the intersection points are found with `Solve`.

```
In[409]:= Plot[{2x^2, 1 + x^2}, {x, -3/2, 3/2},
           PlotStyle- > {GrayLevel[0], GrayLevel[0.3]},
           AspectRatio- > Automatic]
In[410]:= Solve[2x^2 == 1 + x^2]
Out[410]= {{x → -1}, {x → 1}}
```

Using (2.33) and taking advantage of symmetry, the area of R is given by

$$A = \iint_R dA = 2\int_0^1 \int_{2x^2}^{1+x^2} dy\, dx,$$

which we compute with `Integrate`.

```
In[411]:= 2 Integrate[1, {x, 0, 1}, {y, 2x^2, 1 + x^2}]
Out[411]= 4/3
```

We conclude that the area of R is $4/3$. □

If the problem exhibits "circular symmetry," changing to polar coordinates is often useful. If $R = \left\{(r, \theta)\,|\,a \leq r \leq b, \alpha \leq \theta \leq \beta\right\}$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

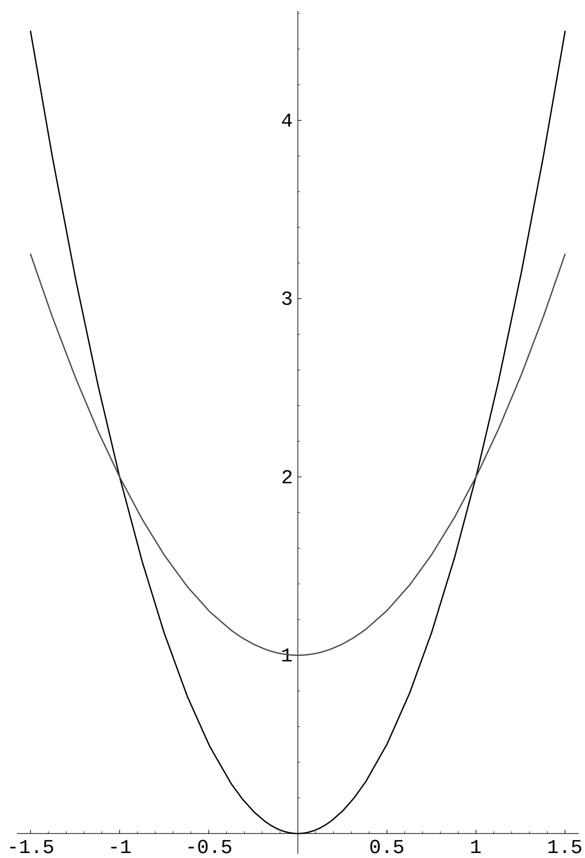


Figure 2.47: $y = 2x^2$ and $y = 1 + x^2$ for $-3/2 \le x \le 3/2$

Example 71. Find the surface area of the portion of

$$f(x,y) = \sqrt{4 - x^2 - y^2}$$

that lies above the region $R = \left\{(x,y) \middle| x^2 + y^2 \le 1\right\}$.

Solution. First, observe that the domain of $f(x,y)$ is

$$\left\{(x,y) \middle| -\sqrt{4-y^2} \le x \le \sqrt{4-y^2}, -2 \le y \le 2\right\} = \{(r,\theta) | 0 \le r \le 2, 0 \le \theta \le 2\pi\}.$$

Similarly,

$$R = \left\{(x,y) \middle| -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}, -1 \le y \le 1\right\} = \{(r,\theta) | 0 \le r \le 1, 0 \le \theta \le 2\pi\}.$$

With this observation, we use `ParametricPlot3D` to graph $f(x,y)$ in `p1` and the portion of the graph of $f(x,y)$ above R in `p2` and show the two graphs together with `Show`. We wish to find the area of the black region in Figure 2.48.

```
In[412]:= f[x_, y_] = Sqrt[4 - x^2 - y^2];

In[413]:= p1 = ParametricPlot3D[{r Cos[t], r Sin[t],
              f[r Cos[t], r Sin[t]]}, {r, 0, 2}, {t, 0, 2π},
            PlotPoints- > 45, DisplayFunction- > Identity];
          p2 = ParametricPlot3D[
            {r Cos[t], r Sin[t], f[r Cos[t], r Sin[t]], GrayLevel[0.3]},
            {r, 0, 1}, {t, 0, 2π},
            PlotPoints- > 45, DisplayFunction- > Identity];
          Show[p1, p2, DisplayFunction- > $DisplayFunction,
            BoxRatios- > Automatic]
```

We compute $f_x(x,y)$, $f_y(x,y)$ and $\sqrt{\left[f_x(x,y)\right]^2 + \left[f_y(x,y)\right]^2 + 1}$ with `D` and `Simplify`.

```
In[414]:= fx = D[f[x, y], x]

          fy = D[f[x, y], y]
```

Out[414]= $-\frac{x}{\sqrt{4 - x^2 - y^2}}$

Out[414]= $-\frac{y}{\sqrt{4 - x^2 - y^2}}$

Then, using (2.34), the surface area is given by

$$\begin{aligned} SA &= \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA \\ &= \iint_R \frac{2}{\sqrt{4 - x^2 - y^2}}\, dA \\ &= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\sqrt{4 - x^2 - y^2}}\, dx\, dy. \end{aligned} \tag{2.36}$$

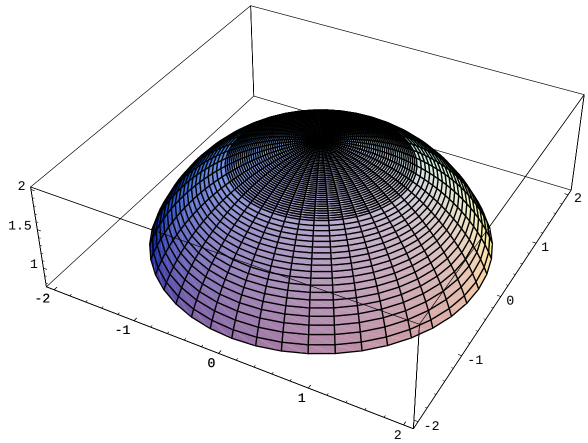


Figure 2.48: The portion of the graph of $f(x, y)$ above R

However, notice that in polar coordinates,

$$R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$$

so in polar coordinates the surface area is given by

$$SA = \int_0^{2\pi} \int_0^1 \frac{2}{\sqrt{4 - r^2}} \, r \, dr \, d\theta,$$

```
In[415]:= s1 = Simplify[Sqrt[1 + fx^2 + fy^2]]
```

$$\text{Out[415]= } 2\sqrt{-\frac{1}{-4 + x^2 + y^2}}$$

```
In[416]:= s2 = Simplify[s1 /. {x- > r Cos[t], y- > r Sin[t]}]
```

$$\text{Out[416]= } 2\sqrt{\frac{1}{4 - r^2}}$$

which is much easier to evaluate than (2.36). We evaluate the iterated integral with `Integrate`

```
In[417]:= s3 = Integrate[r s2, {t, 0, 2π}, {r, 0, 1}]
```

$$\text{Out[417]= } 2\left(4 - 2\sqrt{3}\right)\pi$$

```
In[418]:= N[s3]
Out[418]= 3.36715
```

and conclude that the surface area is $\left(8 - 4\sqrt{3}\right)\pi \approx 3.367$. □

Example 72. Find the volume of the region between the graphs of $z = 4 - x^2 - y^2$ and $z = 2 - x$.

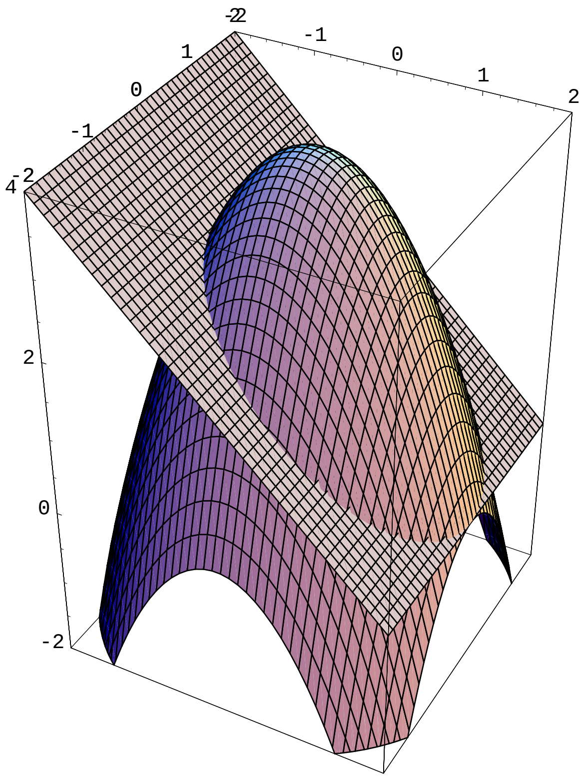


Figure 2.49: $z = 4 - x^2 - y^2$ and $z = 2 - x$ for $-2 \le x \le 2$ and $-2 \le y \le 2$

Solution. We begin by graphing $z = 4 - x^2 - y^2$ and $z = 2 - x$ together with `Plot3D` in Figure 2.49.

```
In[419]:= p1 = Plot3D[4 - x^2 - y^2, {x, -2, 2}, {y, -2, 2},
              PlotPoints- > 40, DisplayFunction- > Identity];
          p2 = Plot3D[2 - x, {x, -2, 2}, {y, -2, 2},
              PlotPoints- > 40, DisplayFunction- > Identity];
          Show[p1, p2, PlotRange- > {{-2, 2}, {-2, 2}, {-2, 4}},
            BoxRatios- > Automatic,
            DisplayFunction- > $DisplayFunction]
```

The region of integration, R, is determined by graphing $4 - x^2 - y^2 = 2 - x$ in Figure 2.50.

```
In[420]:= ContourPlot[4 - x^2 - y^2 - (2 - x), {x, -2, 2}, {y, -2, 2},
            Contours- > {0}, ContourShading- > False, PlotPoints- > 50,
            Frame- > False, Axes- > Automatic, AxesOrigin- > {0, 0}]
```

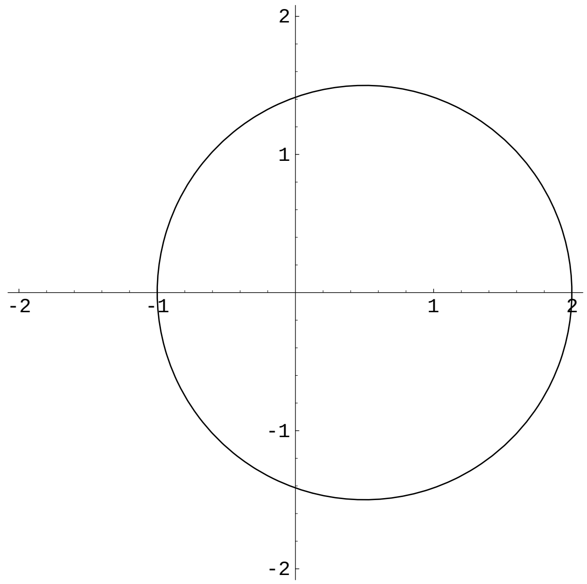


Figure 2.50: Graph of $4 - x^2 - y^2 = 2 - x$

Completing the square shows us that

$$\begin{aligned} R &= \left\{ (x,y) \left| \left(x - \frac{1}{2}\right)^2 + y^2 \le \frac{9}{4} \right. \right\} \\ &= \left\{ (x,y) \left| \frac{1}{2} - \frac{1}{2}\sqrt{9 - 4y^2} \le x \le \frac{1}{2} + \frac{1}{2}\sqrt{9-4y^2}, -\frac{3}{2} \le y \le \frac{3}{2} \right. \right\}. \end{aligned}$$

Thus, using (2.35), the volume of the solid is given by

$$\begin{aligned} V &= \iint_R \left[\left(4 - x^2 - y^2\right) - (2 - x)\right] dA \\ &= \int_{-\frac{3}{2}}^{\frac{3}{2}} \int_{\frac{1}{2}-\frac{1}{2}\sqrt{9-4y^2}}^{\frac{1}{2}+\frac{1}{2}\sqrt{9-4y^2}} \left[\left(4 - x^2 - y^2\right) - (2 - x)\right] dx\, dy, \end{aligned}$$

which we evaluate with `Integrate`.

```
In[421]:= i1 = Integrate[(4 - x^2 - y^2) - (2 - x), {y, -3/2, 3/2},
              {x, 1/2 - 1/2Sqrt[9 - 4y^2], 1/2 + 1/2 Sqrt[9 - 4y^2]}]
```

Out[421]= $\frac{81\ \pi}{32}$

```
In[422]:= N[i1]
Out[422]= 7.95216
```

We conclude that the volume is $\frac{81}{32}\pi \approx 7.952$. □

Triple Iterated Integrals

Triple iterated integrals are calculated in the same manner as double iterated integrals.

Example 73. Evaluate

$$\int_0^{\pi/4}\int_0^{y}\int_0^{y+z}(x+2z)\sin y\,dx\,dz\,dy.$$

Solution. Entering

```
In[423]:= i1 = Integrate[(x + 2z)  Sin[y], {y, 0, π/4}, {z, 0, y},
               {x, 0, y + z}]
```

$$Out[423]= -\frac{17}{\sqrt{2}}+\frac{17\ \pi}{4\ \sqrt{2}}+\frac{17\ \pi^2}{32\ \sqrt{2}}-\frac{17\ \pi^3}{384\ \sqrt{2}}$$

calculates the triple integral exactly with `Integrate`.

An approximation of the exact value is found with `N`.

```
In[424]:= N[i1]
Out[424]= 0.157206
```

□

We illustrate how triple integrals can be used to find the volume of a solid when using spherical coordinates.

Example 74. Find the volume of the torus with equation in spherical coordinates $\rho = \sin\phi$.

Solution. We proceed by graphing the torus with `SphericalPlot3D` in Figure 2.52, which is contained in the `ParametricPlot3D` package that is located in the **Graphics** directory (see Figure 2.51).

```
In[425]:= << Graphics‘ParametricPlot3D‘

          SphericalPlot3D[
            in[ϕ], {ϕ, 0, π}, {θ, 0, 2π}, PlotPoints- > 40]
```

In general, the volume of the solid region D is given by

$$V = \iiint_D dV.$$

Thus, the volume of the torus is given by the triple iterated integral

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta,$$

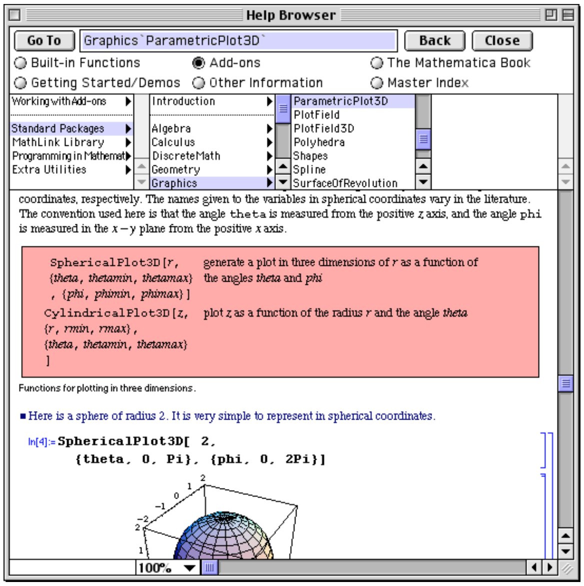


Figure 2.51: Mathematica's help for `SphericalPlot3D`

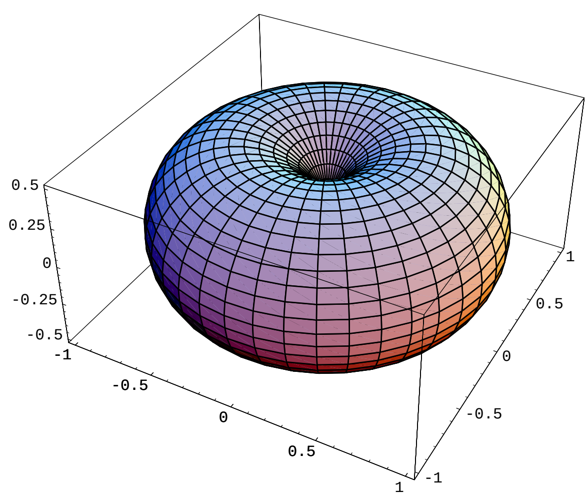


Figure 2.52: A graph of the torus

```
In[426]:= i1 = Integrate[ρ^2 Sin[ϕ], {θ, 0, 2π},
             {ϕ, 0, π}, {ρ, 0, Sin[ϕ]}]
Out[426]= π^2/4
In[427]:= N[i1]
Out[427]= 2.4674
```

which we evaluate with `Integrate`. We conclude that the volume of the torus is $\frac{1}{4}\pi^2 \approx 2.467$. □

2.6 Vector Calculus

2.6.1 Basic Operations on Vectors

We review the elementary properties of vectors in in space. Let

$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

and

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

be vectors in space.

In space, the **standard unit vectors** are $\mathbf{i} = \langle 1,0,0\rangle$, $\mathbf{j} = \langle 0,1,0\rangle$, and $\mathbf{k} = \langle 0,0,1\rangle$. With the exception of the cross product, the vector operations discussed here are performed in the same way for vectors in the plane as they are in space. In the plane, the **standard unit vectors** are $\mathbf{i} = \langle 1,0\rangle$ and $\mathbf{j} = \langle 0,1\rangle$.

1. $\mathbf{u}$ and $\mathbf{v}$ are **equal** if and only if their components are equal:

$$\mathbf{u} = \mathbf{v} \Leftrightarrow u_1 = v_1, u_2 = v_2, \text{ and } u_3 = v_3.$$

2. The **length** (or **norm**) of $\mathbf{u}$ is

$$\|\mathbf{u}\| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}.$$

3. If c is a scalar (number),

$$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle.$$

4. The **sum** of $\mathbf{u}$ and $\mathbf{v}$ is defined to be the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle.$$

5. If $\mathbf{u} \neq \mathbf{0}$, a unit vector with the same direction as $\mathbf{u}$ is

A **unit vector** is a vector with length 1.

$$\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}} \langle u_1, u_2, u_3 \rangle.$$

6. $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if there is a scalar c so that $\mathbf{u} = c\mathbf{v}$.

7. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Consequently, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u}\cdot\mathbf{v} = 0$.

8. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}\\ &= \left(u_2v_3 - u_3v_2\right)\mathbf{i} - \left(u_1v_3 - u_3v_1\right)\mathbf{j} + \left(u_1v_2 - u_2v_1\right)\mathbf{k}.\end{aligned}$$

You should verify that $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0$ and $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 0$. Hence, $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Topics from linear algebra (including determinants) are discussed in more detail in the next chapter. For now, we illustrate several of the basic operations listed above. In Mathematica, many vector calculations take advantage of functions contained in the `VectorAnalysis` package located in the **Calculus** directory. Use Mathematica's help facility to obtain general help regarding the `VectorAnalysis` package as shown in Figure 2.53.

Example 75. Let $\mathbf{u} = \langle 3, 4, 1\rangle$ and $\mathbf{v} = \langle -4, 3, -2\rangle$. Calculate (a) $\mathbf{u}\cdot\mathbf{v}$, (b) $\mathbf{u}\times\mathbf{v}$, (c) $\|\mathbf{u}\|$, and (d) $\|\mathbf{v}\|$. (e) Find the angle between $\mathbf{u}$ and $\mathbf{v}$. (f) Find unit vectors with the same direction as $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$.

Solution. After loading the `VectorAnalysis` package, we define $\mathbf{u} = \langle 3, 4, 1\rangle$ and $\mathbf{v} = \langle -4, 3, -2\rangle$. Notice that to define $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ with Mathematica, we use the form

`u={u1,u2,u3}`.

Similarly, to define $\mathbf{u} = \langle u_1, u_2\rangle$, we use the form `u={u1,u2}`.

We illustrate the use of `DotProduct` and `CrossProduct`, both of which are contained in the `VectorAnalysis` package, to calculate (a)-(d).

Remark. Generally, `u.v` returns the same result as `DotProduct[u,b]`.

```
In[428]:= << Calculus`VectorAnalysis`

In[429]:= u = {3, 4, 1};
          v = {-4, 3, -2};

In[430]:= udv = DotProduct[u, v]
Out[430]= -2
```

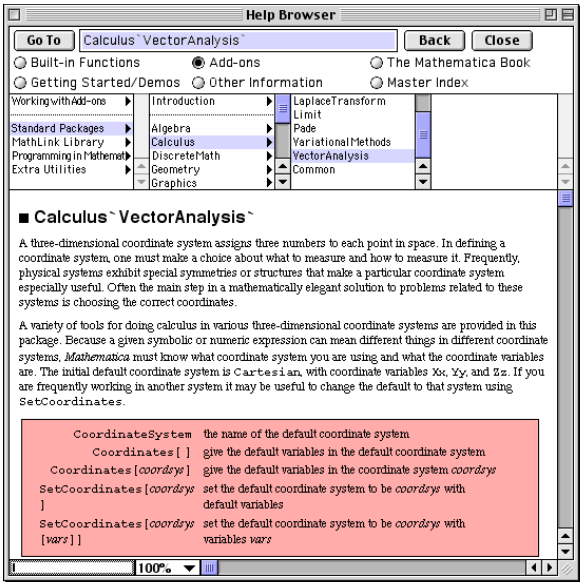


Figure 2.53: Mathematica's help for the `VectorAnalysis` package

```
In[431]:= ucv = CrossProduct[u, v]
Out[431]= {-11, 2, 25}

In[432]:= ν = Sqrt[u.u]
Out[432]= √26

In[433]:= nv = Sqrt[v.v]
Out[433]= √29
```

We use the formula $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$ to find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

```
In[434]:= ArcCos[u.v/(ν nv)]

          N[%]
```

Out[434]= $\text{ArcCos}\left[-\sqrt{\frac{2}{377}}\right]$

```
Out[434]= 1.6437
```

Unit vectors with the same direction as $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$ are found next.

```
In[435]:= normu = u/ν

          normv = v/nv

          nucrossv = ucv/Sqrt[ucv.ucv]
```

Out[435]= $\left\{\frac{3}{\sqrt{26}}, 2\sqrt{\frac{2}{13}}, \frac{1}{\sqrt{26}}\right\}$

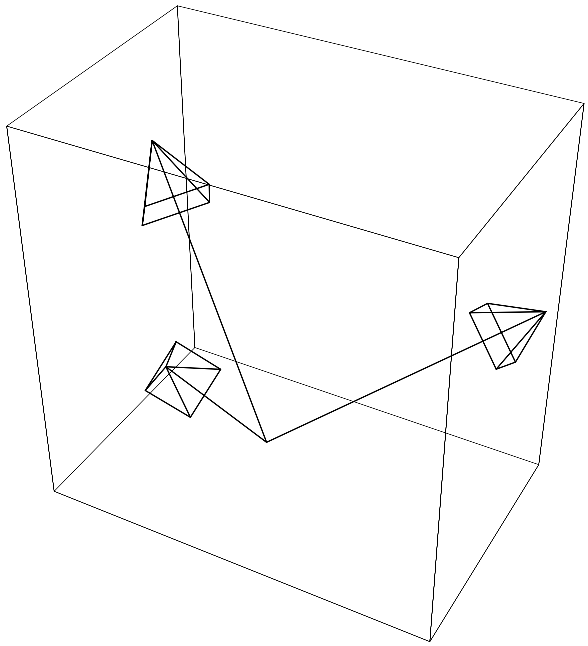


Figure 2.54: Orthogonal vectors

```
Out[435]= {-4/√29, 3/√29, -2/√29}
```

$$\left\{-\frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}}, -\frac{2}{\sqrt{29}}\right\}$$

```
Out[435]=
```

$$\left\{-\frac{11}{5\sqrt{30}}, \frac{\sqrt{\frac{2}{15}}}{5}, \sqrt{\frac{5}{6}}\right\}$$

We can graphically confirm that these three vectors are orthogonal by graphing all three vectors with the `ListPlotVectorField3D` function, which is contained in the `PlotField3D` package. After loading the `PlotField3D` package, the command

```
ListPlotVectorField3D[listofvectors]
```

graphs the list of vectors `listofvectors`. Each element of `listofvectors` is of the form `{{u1,u2,u3},{v1,v2,v3}}` where (u_1, u_2, u_3) and (v_1, v_2, v_3) are the initial and terminal points of each vector. We show the vectors in Figure 2.54.

```
In[436]:= << Graphics`PlotField3D`
```

```
In[437]:= ListPlotVectorField3D[{{{0, 0, 0}, normu},
              {{0, 0, 0}, normv}, {{0, 0, 0}, nucrossv}},
            VectorHeads- > True]
```

In the plot, the vectors do appear to be orthogonal as expected. □

With the exception of the cross product, the calculations described above can also be performed on vectors in the plane.

Example 76. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, the **projection** of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}.$$

Find $\text{proj}_{\mathbf{v}}\mathbf{u}$ if $\mathbf{u} = \langle -1, 4\rangle$ and $\mathbf{v} = \langle 2, 6\rangle$.

Solution. We define $\mathbf{u} = \langle -1, 4\rangle$ and $\mathbf{v} = \langle 2, 6\rangle$.

and then compute $\text{proj}_{\mathbf{v}}\mathbf{u}$.

```
In[438]:= u = {-1, 4};
          v = {2, 6};
          projvu = u.v v/v.v
```

$$\text{Out[438]=}\ \left\{\frac{11}{10}, \frac{33}{10}\right\}$$

Finally, we graph $\mathbf{u}$, $\mathbf{v}$, and $\text{proj}_{\mathbf{v}}\mathbf{u}$ together using `Arrow` and `Show` in Figure 2.55.

```
In[439]:= << Graphics‘Arrow‘

In[440]:= ?Arrow

"Arrow[start, finish, (opts)]isagraphics
   primitiverepresentinganarrowstartingat
   startandendingatfinish."

In[441]:= p1 = Show[Graphics[
               {Arrow[{0, 0}, u], Arrow[{0, 0}, v], Thickness[0.03],
                 Arrow[{0, 0}, projvu, HeadScaling- > Relative]}],
             Axes- > Automatic, AspectRatio- > Automatic,
             DisplayFunction- > Identity];

In[442]:= p2 = Show[Graphics[{Arrow[{0, 0}, u],
                 Arrow[{0, 0}, v], Thickness[0.03], Arrow[{0, 0},
                   projvu, HeadScaling- > Relative], GrayLevel[0.4],
                 Arrow[projvu, u, HeadScaling- > Relative]}],
             Axes- > Automatic, AspectRatio- > Automatic,
             DisplayFunction- > Identity];

In[443]:= Show[GraphicsArray[{p1, p2}]]
```

In the graph, notice that $\mathbf{u} = \text{proj}_{\mathbf{v}}\mathbf{u} + \left(\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}\right)$ and the vector $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is perpendicular to $\mathbf{v}$. □

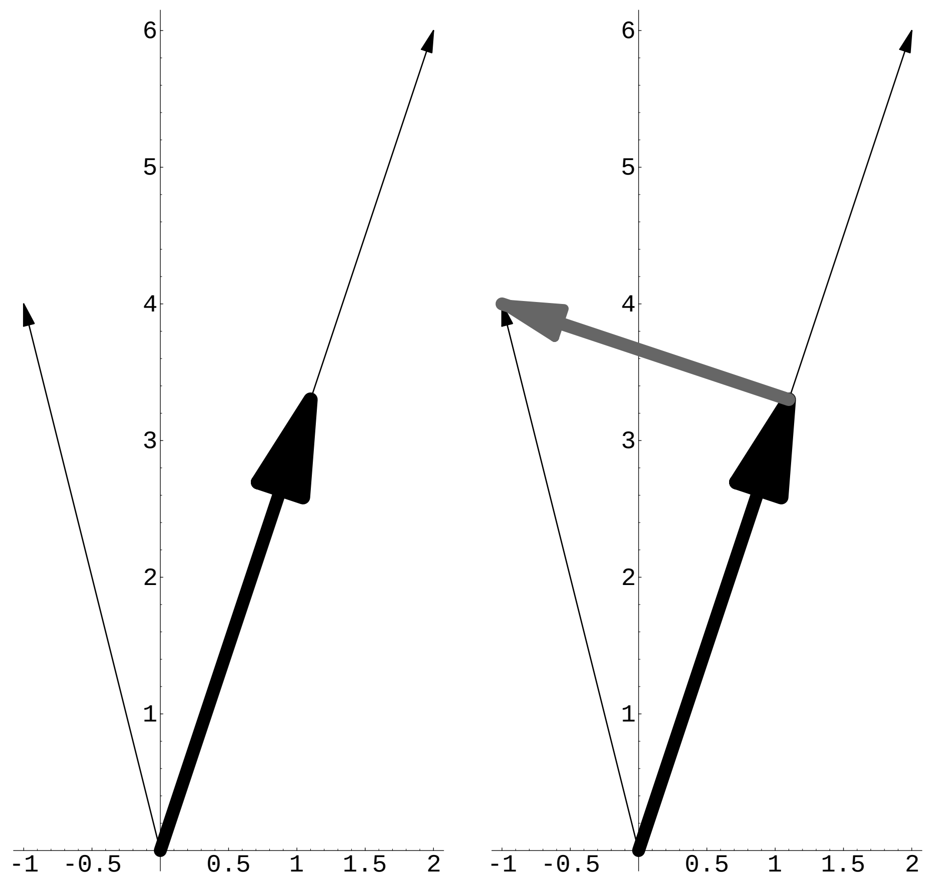


Figure 2.55: Projection of a vector

2.6.2 Vector-Valued Functions

We now turn our attention to vector-valued functions. In particular, we consider vector-valued functions of the following forms.

$$\begin{aligned} \text{Plane curves:} \quad & \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} && (2.37) \\ \text{Space curves:} \quad & \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} && (2.38) \\ \text{Parametric surfaces:} \quad & \mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k} && (2.39) \\ \text{Vector fields in the plane:} \quad & \mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} && (2.40) \\ \text{Vector fields in space:} \quad & \mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k} && (2.41) \end{aligned}$$

For the vector-valued functions (2.37) and (2.38), differentiation and integration is carried out term-by-term, provided that all the terms are differentiable and integrable. Suppose that C is a smooth curve defined by $\mathbf{r}(t)$, $a \le t \le b$.

1. If $\mathbf{r}'(t) \neq \mathbf{0}$, the **unit tangent vector**, $\mathbf{T}(t)$, at t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

2. If $\mathbf{T}'(t) \neq \mathbf{0}$, the **principal unit normal vector**, $\mathbf{N}(t)$, is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

3. The **arc length function**, $s(t)$, is

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\, du.$$

In particular, the length of C on the interval $[a,b]$ is $\int_a^b \|\mathbf{r}'(t)\|\, dt$.

4. The **curvature**, κ, of C at t is

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{a}(t)\cdot\mathbf{N}(t)}{\|\mathbf{v}(t)\|^2} = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3},$$

where $\mathbf{v}(t) = \mathbf{r}'(t)$ and $\mathbf{a}(t) = \mathbf{r}''(t)$

It is a good exercise to show that the curvature of a circle of radius r is $1/r$.

Example 77 (Folium of Descartes). Consider the **Folium of Descartes**,

$$\mathbf{r}(t) = \frac{3at}{1+t^3}\mathbf{i} + \frac{3at^2}{1+t^3}\mathbf{j}$$

for $t \neq -1$, if $a = 1$. (a) Find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$ and $\int \mathbf{r}(t)\, dt$. (b) Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$. (c) Find the curvature, κ. (d) Find the length of the loop of the Folium.

Solution. (a) After defining $\mathbf{r}(t)$,

```
In[444]:= r[t_] = {3 a t/(1 + t^3), 3 a t^2/(1 + t^3)};
          a = 1;
```

we compute $\mathbf{r}'(t)$ and $\int \mathbf{r}(t)\,dt$ with `'`, `''` and `Integrate`, respectively. We name $\mathbf{r}'(t)$ `dr`, $\mathbf{r}''(t)$ `dr2`, and $\int \mathbf{r}(t)\,dt$ `ir`.

```
In[445]:= dr = Simplify[r'[t]]

          dr2 = Simplify[r''[t]]

          ir = Integrate[r[t], t]
```

Out[445]= $\left\{\frac{3-6 t^3}{(1+t^3)^2}, -\frac{3 t (-2+t^3)}{(1+t^3)^2}\right\}$

Out[445]= $\left\{\frac{18 t^2 (-2+t^3)}{(1+t^3)^3}, \frac{6 (1-7 t^3+t^6)}{(1+t^3)^3}\right\}$

Out[445]= $\left\{\sqrt{3}\ \mathrm{ArcTan}\left[\frac{-1+2 t}{\sqrt{3}}\right] - \mathrm{Log}[1+t] + \frac{1}{2} \mathrm{Log}\left[1-t+t^2\right], \mathrm{Log}\left[1+t^3\right]\right\}$

(b) Mathematica does not automatically make assumptions regarding the value of t, so does not algebraically simplify $\|\mathbf{r}'(t)\|$ as we might typically do unless we use `PowerExpand`.

`PowerExpand[Sqrt[x^2]]` returns x

```
In[446]:= nr = PowerExpand[Sqrt[dr.dr]//Simplify]
```

Out[446]= $\frac{3 \sqrt{1+4 t^2-4 t^3-4 t^5+4 t^6+t^8}}{(1+t^3)^2}$

The unit tangent vector, $\mathbf{T}(t)$ is formed in `ut`.

```
In[447]:= ut = 1/nr dr//Simplify
```

Out[447]= $\left\{\frac{1-2 t^3}{\sqrt{1+4 t^2-4 t^3-4 t^5+4 t^6+t^8}}, -\frac{t (-2+t^3)}{\sqrt{1+4 t^2-4 t^3-4 t^5+4 t^6+t^8}}\right\}$

We perform the same steps to compute the unit normal vector, $\mathbf{N}(t)$. In particular, note that `dutb` = $\|\mathbf{T}'(t)\|$.

```
In[448]:= dut = D[ut, t]//Simplify
```

Out[448]= $\left\{\frac{2 t (-2+t^3) (1+t^3)^2}{(1+4 t^2-4 t^3-4 t^5+4 t^6+t^8)^{3/2}}, -\frac{2 (-1+3 t^6+2 t^9)}{(1+4 t^2-4 t^3-4 t^5+4 t^6+t^8)^{3/2}}\right\}$

```
In[449]:= duta = dut.dut//Simplify
```

Out[449]= $\frac{4 (1+t^3)^4}{(1+4 t^2-4 t^3-4 t^5+4 t^6+t^8)^2}$

```
In[450]:= dutb = PowerExpand[Sqrt[duta]]
```

Out[450]= $\frac{2 (1+t^3)^2}{1+4 t^2-4 t^3-4 t^5+4 t^6+t^8}$

```
In[451]:= nt = 1/dutb dut//Simplify
```

Out[451]= $\left\{\frac{t\,(-2+t^3)}{\sqrt{1+4\,t^2-4\,t^3-4\,t^5+4\,t^6+t^8}},\ \frac{1-2\,t^3}{\sqrt{1+4\,t^2-4\,t^3-4\,t^5+4\,t^6+t^8}}\right\}$

(c) We use the formula $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$ to determine the curvature in `curvature`.

```
In[452]:= curvature = Simplify[dutb/nr]
```

Out[452]= $\frac{2\,(1+t^3)^4}{3\,(1+4\,t^2-4\,t^3-4\,t^5+4\,t^6+t^8)^{3/2}}$

We graphically illustrate the unit tangent and normal vectors at $\mathbf{r}(1) = \langle 3/2, 3/2\rangle$. First, we compute the unit tangent and normal vectors if $t = 1$ using `/.`.

```
In[453]:= ut1 = ut/.t- > 1
```

Out[453]= $\left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$

```
In[454]:= nt1 = nt/.t- > 1
```

Out[454]= $\left\{-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$

We then compute the curvature if $t = 1$ in `smallk`. The center of the osculating circle at $\mathbf{r}(1)$ is found in `x0` and `y0`.

The radius of the osculating circle is $1/\kappa$; the position vector of the center is $\mathbf{r} + \frac{1}{\kappa}\mathbf{N}$.

```
In[455]:= smallk = curvature/.t- > 1

          N[smallk]

          N[1/smallk]

          x0 = r[t][[1]] - dr.dr
                 r[[2]]/(dr[[1]]dr2[[2]] - dr2[[1]]dr[[2]])/.t- > 1

          y0 = r[t][[2]]-
               r.dr dr[[2]]/(dr[[1]]dr2[[2]] - dr2[[1]]dr[[2]])/.t
             - > 1
```

Out[455]= $\frac{8\sqrt{2}}{3}$

Out[455]= 3.77124

Out[455]= 0.265165

Out[455]= $\frac{21}{16}$

Out[455]= $\frac{21}{16}$

We now load the `Arrow` package and graph $\mathbf{r}(t)$ with `ParametricPlot`. The unit tangent and normal vectors at $\mathbf{r}(1)$ are graphed with `arrow` in `a1` and `a2`. The osculating circle at $\mathbf{r}(1)$ is graphed with `Circle` in `c1`. All four graphs are displayed together with `Show` in Figure 2.56.

`Graphics[Circle[{x0, y0}, r]]` is a two-dimensional graphics object that represents a circle of radius r centered at the point (x_0, y_0). Use `Show` to display the graph.

```
In[456]:= << Graphics‘Arrow‘
```

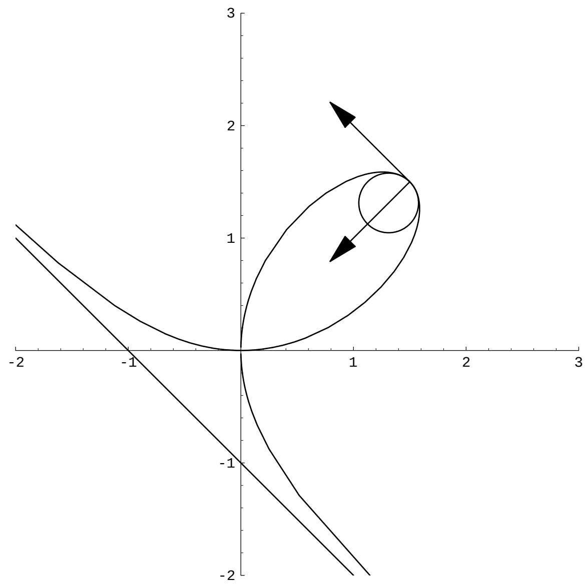


Figure 2.56: The Folium with an osculating circle

```
In[457]:= p1 = ParametricPlot[Evaluate[r[t]], {t, -100, 100},
              PlotRange-> {{-2, 3}, {-2, 3}}, PlotPoints-> 200,
              AspectRatio-> 1, DisplayFunction-> Identity];
          p2 = Show[Graphics[{Circle[{x0, y0}, 1/smallk],
                 Arrow[r[1], r[1] + ut1], Arrow[r[1], r[1] + nt1]}],
              DisplayFunction-> Identity];
          Show[p1, p2, DisplayFunction-> $DisplayFunction]
```

(d) The loop is formed by graphing $\mathbf{r}(t)$ for $t \geq 0$. Hence, the length of the loop is given by the improper integral $\int_0^\infty \|\mathbf{r}(t)\|\, dt$, which we compute with `NIntegrate`.

```
In[458]:= NIntegrate[nr, {t, 0, ∞}]
Out[458]= 4.91749
```

□

Recall that the **gradient** of $z = f(x, y)$ is the vector-valued function $\nabla f(x, y) = \left\langle f_x(x, y),\, f_y(x, y)\right\rangle$. Similarly, we define the **gradient** of $w = f(x, y, z)$ to be

$$\nabla f(x, y, z) = \left\langle f_x(x, y, z),\, f_y(x, y, z),\, f_z(x, y, z)\right\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}. \tag{2.42}$$

A vector field $\mathbf{F}$ is **conservative** if there is a function f, called a **potential function**, satisfying $\nabla f = \mathbf{F}$. In the special case that $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

The **divergence** of the the vector field $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is the scalar field

$$\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{2.43}$$

The `Div` command, which is contained in the `VectorAnalysis` package, can be used to find the divergence of a vector field:

```
Div[{P(x,y,z),Q(x,y,z),R(x,y,z)},Cartesian[x,y,z]]
```

computes the divergence of $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$. The **laplacian** of the scalar field $w = f(x, y, z)$ is defined to be

$$\text{div}\,(\nabla f) = \nabla \cdot (\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \triangle f. \tag{2.44}$$

In the same way that `Div` computes the divergence of a vector field, `Laplacian`, which is also contained in the `VectorAnalysis` package, computes the laplacian of a scalar field.

The **curl** of the vector field $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is

$$\begin{aligned} \text{curl}\mathbf{F}(x, y, z) &= \nabla \times \mathbf{F}(x, y, z) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y, z) & Q(x, y, z) & R(x, y, z) \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}. \end{aligned} \tag{2.45}$$

If $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$, $\mathbf{F}$ is conservative if and only if $\text{curl}\mathbf{F}(x, y, z) = \mathbf{0}$, in which case $\mathbf{F}$ is said to be **irrotational**.

Example 78. Determine if

$$\mathbf{F}(x, y) = \left(1 - 2x^2\right) y e^{-x^2 - y^2}\mathbf{i} + \left(1 - 2y^2\right) x e^{-x^2 - y^2}\mathbf{j}$$

is conservative. If $\mathbf{F}$ is conservative find a potential function for $\mathbf{F}$.

Solution. We define $P(x, y) = \left(1 - 2x^2\right) y e^{-x^2 - y^2}$ and $Q(x, y) = \left(1 - 2y^2\right) x e^{-x^2 - y^2}$. Then we use `D` and `Simplify` to see that $P_y(x, y) = Q_x(x, y)$. Hence, $\mathbf{F}$ is conservative.

```
In[459]:= p[x_, y_] = (1 - 2x^2)y Exp[-x^2 - y^2];
          q[x_, y_] = (1 - 2y^2)x Exp[-x^2 - y^2];

In[460]:= Simplify[D[p[x, y], y]]

          Simplify[D[q[x, y], x]]
Out[460]= e^(-x^2-y^2) (-1 + 2 x^2) (-1 + 2 y^2)
```

Out[460]= $e^{-x^2-y^2}\left(-1+2\ x^2\right)\left(-1+2\ y^2\right)$

We use `Integrate` to find f satisfying $\nabla f = \mathbf{F}$.

```
In[461]:= i1 = Integrate[p[x, y], x] + g[y]
Out[461]= e^(-x^2-y^2) x y + g[y]

In[462]:= Solve[D[i1, y] == q[x, y], g'[y]]
Out[462]= BoxData({{g'[y] → 0}})

In[463]:= f = i1/.g[y]- > 0
Out[463]= e^(-x^2-y^2) x y
```

Remember that the vectors $\mathbf{F}$ are perpendicular to the level curves of f. To see this, we normalize $\mathbf{F}$ in `uv`.

```
In[464]:= uv = {p[x, y], q[x, y]}/
             Sqrt[{p[x, y], q[x, y]}.{p[x, y], q[x, y]}]//
           Simplify
```

Out[464]= $$\left\{-\frac{e^{-x^2-y^2}\ (-1+2\ x^2)\ y}{\sqrt{e^{-2\ \left(x^2+y^2\right)}\ (y^2+4\ x^4\ y^2+x^2\ (1-8\ y^2+4\ y^4))}},\right.$$
$$\left.-\frac{e^{-x^2-y^2}\ x\ (-1+2\ y^2)}{\sqrt{e^{-2\ \left(x^2+y^2\right)}\ (y^2+4\ x^4\ y^2+x^2\ (1-8\ y^2+4\ y^4))}}\right\}$$

See Figure 4.20.

We then graph several level curves of f in `cp` with `ContourPlot` and several vectors of `uv` with `PlotVectorField`, which is contained in the `PlotField` package, in `fp`. We show the graphs together with `Show` in Figure 2.57.

```
In[465]:= << Graphics`PlotField`

          cp = ContourPlot[f, {x, -3/2, 3/2}, {y, -3/2, 3/2},
             ontours- > 15, ContourShading- > False, PlotPoints- > 60, D
             isplayFunction- > Identity];

In[466]:= fp = PlotVectorField[uv, {x, -3/2, 3/2}, {y, -3/2, 3/2},
             DisplayFunction- > Identity];
Power :: "infy" :
  "Infiniteexpression1⁄0 encountered."
Power :: "infy" :
  "Infiniteexpression1⁄0 encountered."

In[467]:= Show[cp, fp, DisplayFunction- > $DisplayFunction]
```

Note that we can use `PlotGradientField`, which is contained in the `PlotField` package, to graph several vectors of ∇f. However, the vectors are scaled and it can be difficult to see that the vectors are perpendicular to the level curves of f. The advantage of proceeding this way is that by graphing unit vectors, it is easier to see that the vectors are perpendicular to the level curves of f in the resulting plot. □

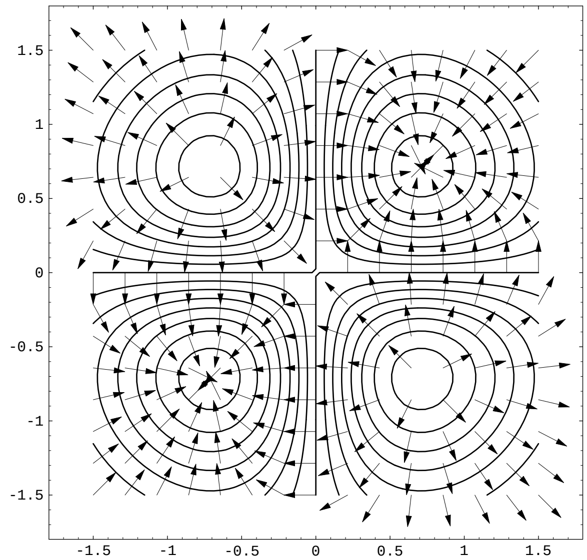


Figure 2.57: The vectors **F** are perpendicular to the level curves of f

Example 79. (a) Show that

$$\mathbf{F}(x,y,z) = -10xy^2\mathbf{i} + \left(3z^3 - 10x^2y\right)\mathbf{j} + 9yz^2\mathbf{k}$$

is irrotational. (b) Find f satisfying $\nabla f = \mathbf{F}$. (c) Compute div $\mathbf{F}$ and $\nabla^2 f$.

Solution. (a) After defining $\mathbf{F}(x,y,z)$, we use `Curl`, which is contained in the `VectorAnalysis` package, to see that curl $\mathbf{F}(x,y,z) = \mathbf{0}$.

```
In[468]:= << Calculus`VectorAnalysis`

In[469]:= BoxData({Clear[f],  f[x_, y_, z_] = {-10x y^2, 3z^3-10x^2 y, 9 y z ^2}})
Out[469]= {-10 x y^2, -10 x^2 y + 3 z^3, 9 y z^2}

In[470]:= Curl[f[x, y, z]]
Out[470]= {0, 0, 0}
```

(b) We then use `Integrate` to find $w = f(x,y,z)$ satisfying $\nabla f = \mathbf{F}$.

```
In[471]:= i1 = Integrate[f[x, y, z][[1]], x] + g[y, z]
Out[471]= -5 x^2 y^2 + g[y, z]

In[472]:= i2 = D[i1, y]
Out[472]= BoxData(-10 x^2 y + g^(1,0)[y, z])

In[473]:= BoxData(Solve[i2 == f[x, y, z][[2]], g^(1,0)[y, z]])
Out[473]= BoxData({{g^(1,0)[y, z] → 3 z^3}})
```

```
In[474]:= i3 = Integrate[3z^3, y] + h[z]
Out[474]= 3 y z^3 + h[z]

In[475]:= i4 = i1/.g[y, z]-> i3
Out[475]= -5 x^2 y^2 + 3 y z^3 + h[z]

In[476]:= Solve[D[i4, z] == f[x, y, z][[3]]]
Out[476]= BoxData({{h'[z] → 0}})

In[477]:= lf = -5 x^2 y^2 + 3 y z^3;
```

∇f is orthogonal to the level surfaces of f. To illustrate this, we use ContourPlot3D, which is contained in the ContourPlot3D package, to graph the level surface of $w = f(x, y, z)$ corresponding to $w = -1$ for $-2 \le x \le 2$, $-2 \le y \le 2$, and $-2 \le z \le 2$ in pf. We then use PlotGradientField3D, which is contained in the PlotField3D package, to graph several vectors in the gradient field of f over the same domain in gradf. The two plots are shown together with Show in Figure 2.58. In the plot, notice that the vectors appear to be perpendicular to the surface.

```
In[478]:= << Graphics`PlotField3D`

          << Graphics`ContourPlot3D`

In[479]:= pf = ContourPlot3D[lf, {x, -2, 2}, {y, -2, 2}, {z, -2, 2},
             PlotPoints-> {5, 7}, DisplayFunction-> Identity];

In[480]:= gf = PlotGradientField3D[lf, {x, -2, 2}, {y, -2, 2},
             {z, -2, 2}, DisplayFunction-> Identity];

In[481]:= Show[pf, gf, DisplayFunction-> $DisplayFunction]
```

For (c), we take advantage of Div and Laplacian. As expected, the results are the same.

```
In[482]:= Div[f[x, y, z], Cartesian[x, y, z]]
Out[482]= -10 x^2 - 10 y^2 + 18 y z

In[483]:= Laplacian[lf, Cartesian[x, y, z]]
Out[483]= -10 x^2 - 10 y^2 + 18 y z
```

□

2.6.3 Line Integrals

If $\mathbf{F}$ is continuous on the smooth curve C with parametrization $\mathbf{r}(t)$, $a \le t \le b$, the **line integral** of $\mathbf{F}$ on C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt \tag{2.46}$$

If $\mathbf{F}$ is conservative and C is piecewise smooth, line integrals can be evaluated using the *Fundamental Theorem of Line Integrals*.

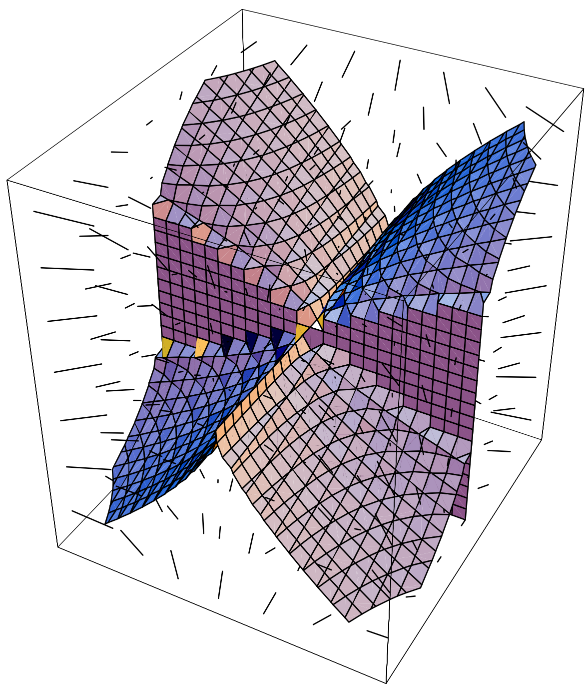


Figure 2.58: ∇f is orthogonal to the level surfaces of f

Theorem 18 (Fundamental Theorem of Line Integrals). *If* $\mathbf{F}$ *is conservative and the curve* C *defined by* $\mathbf{r}(t)$*,* $a \le t \le b$ *is piecewise smooth,*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{2.47}$$

where $\mathbf{F} = \nabla f$*.*

Example 80. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = (e^{-y} - ye^{-x})\,\mathbf{i} + (e^{-x} - xe^{-y})\,\mathbf{i}$ and C is defined by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \ln(2t/\pi)\,\mathbf{j}$, $\pi/2 \le t \le 4\pi$.

Solution. We see that $\mathbf{F}$ is conservative with `D` and find that $f(x, y) = xe^{-y} + ye^{-x}$ satisfies $\nabla f = \mathbf{F}$ with `Integrate`.

```
In[484]:= f[x_, y_] = {Exp[-y] - y Exp[-x], Exp[-x] - x Exp[-y]};
          r[t_] = {Cos[t], Log[2t/π]};

In[485]:= BoxData({D[f[x, y][[1]], y]//Simplify, D[f[x, y][[2]], x]//Simplify })
Out[485]= -e^-x - e^-y
Out[485]= -e^-x - e^-y

In[486]:= lf = Integrate[f[x, y][[1]], x]
Out[486]= e^-y x + e^-x y
```

Hence, using (2.47),

$$\int_C \mathbf{F} \cdot d\mathbf{r} = (xe^{-y} + ye^{-x})\Big]_{x=0,y=0}^{x=1,y=\ln 8} = \frac{3\ln 2}{e} + \frac{1}{8} \approx 0.890.$$

```
In[487]:= xr[t_] = Cos[t];
          yr[t_] = Log[2 t/π];
          {xr[π/2], yr[π/2]}

          {xr[4π], yr[4π]}
Out[487]= {0, 0}
Out[487]= {1, Log[8]}

In[488]:= Simplify[lf/.{x- > 1, y- > Log[8]}]

          N[%]
Out[488]= 1/8 + Log[8]/e
Out[488]= 0.889984
```

□

If C is a piecewise smooth simple closed curve and $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives, *Green's theorem* relates the line integral $\oint_C (P(x, y)\,dx + Q(x, y)\,dy)$ to a double integral.

We assume that the symbol $\oint$ means to evaluate the integral in the positive (or counterclockwise) direction.

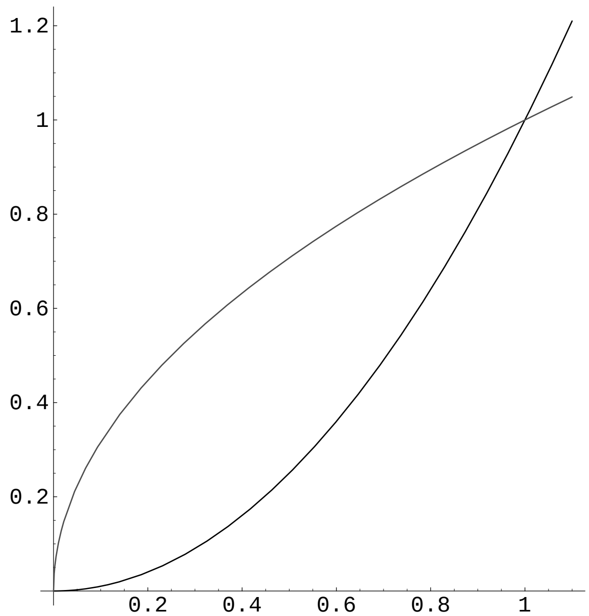


Figure 2.59: $y = x^2$ and $y = \sqrt{x}$, $0 \le x \le 1$

Theorem 19 (Green's Theorem). *Let C be a piecewise smooth simple closed curve in the plane and R the region bounded by C. If* $P(x,y)$ *and* $Q(x,y)$ *have continuous partial derivatives on R,*

$$\oint_C (P(x,y)\,dx + Q(x,y)\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{2.48}$$

Example 81. Evaluate

$$\oint_C (e^x - \sin y)\ dx + (\cos x - e^{-y})\ dy$$

where C is the boundary of the region between $y = x^2$ and $x = y^2$.

Solution. After defining $P(x,y) = e^x - \sin y$ and $Q(x,y) = \cos x - e^{-y}$, we use `Plot` to determine the region R bounded by C in Figure 2.59.

```
In[489]:= p[x_, y_] = Exp[-x] - Sin[y];
          q[x_, y_] = Cos[x] - Exp[-y];
          Plot[{x^2, Sqrt[x]}, {x, 0, 1.1},
            PlotStyle-> {GrayLevel[0], GrayLevel[0.3]},
            AspectRatio-> Automatic]
```

Using (2.48),

$$\begin{aligned}\oint_C (e^x - \sin y)\ dx + (\cos x - e^{-y})\ dy &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= \iint_R (\cos y - \sin x)\, dA \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} (\cos y - \sin x)\ dy\, dx,\end{aligned}$$

```
In[490]:= dqdp = Simplify[D[q[x, y], x] - D[p[x, y], y]]
Out[490]= Cos[y] - Sin[x]
```

which we evaluate with `Integrate`.

```
In[491]:= Integrate[dqdp, {x, 0, 1}, {y, x^2, Sqrt[x]}]

          N[%]
```

$$\texttt{Out[491]=}\ -2 - \sqrt{\frac{\pi}{2}}\ \texttt{FresnelC}\left[\sqrt{\frac{2}{\pi}}\right] - \sqrt{\frac{\pi}{2}}\ \texttt{FresnelS}\left[\sqrt{\frac{2}{\pi}}\right] + 4\ \texttt{Sin[1]}$$

```
Out[491]= 0.151091
```

Notice that the result is given in terms of the `FresnelS` and `FresnelC` functions, which are defined by

$$\texttt{FresnelS[x]} = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt \quad \text{and} \quad \texttt{FresnelC[x]} = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt.$$

A more meaningful approximation is obtained with `N`. We conclude that $\int_0^1 \int_{x^2}^{\sqrt{x}} (\cos y - \sin x)\ dy\, dx \approx 0.151$. □

2.6.4 Surface Integrals

Let S be the graph of $z = f(x, y)$ ($y = h(x, z)$, $x = k(y, z)$) and let R_{xy} (R_{xz}, R_{yz}) be the projection of S onto the xy (xz, yz) plane. Then,

$$\iint_S g(x,y,z)\, dS = \iint_{R_{xy}} g\left(x, y, f(x,y)\right) \sqrt{\left[f_x(x,y)\right]^2 + \left[f_y(x,y)\right]^2 + 1}\, dA \qquad (2.49)$$

$$= \iint_{R_{xz}} g\left(x, h(x,z), z\right) \sqrt{\left[h_x(x,z)\right]^2 + \left[h_z(x,z)\right]^2 + 1}\, dA \qquad (2.50)$$

$$= \iint_{R_{xy}} g\left(k(y,z), y, z\right) \sqrt{\left[k_y(y,z)\right]^2 + \left[k_z(y,z)\right]^2 + 1}\, dA. \qquad (2.51)$$

If S is defined parametrically by

$$\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}, \quad (s,t) \in R$$

the formula

$$\iint_S g(x,y,z)\,dS = \iint_R g\left(\mathbf{r}(s,t)\right)\left\|\mathbf{r}_s \times \mathbf{r}_t\right\|\,dA, \tag{2.52}$$

where

$$\mathbf{r}_s = \frac{\partial x}{\partial s}\mathbf{i} + \frac{\partial y}{\partial s}\mathbf{j} + \frac{\partial z}{\partial s}\mathbf{k} \qquad \text{and} \qquad \mathbf{r}_t = \frac{\partial x}{\partial t}\mathbf{i} + \frac{\partial y}{\partial t}\mathbf{j} + \frac{\partial z}{\partial t}\mathbf{k},$$

is also useful.

Theorem 20 (The Divergence Theorem). *Let Q be any domain with the property that each line through any interior point of the domain cuts the boundary in exactly two points, and such that the boundary S is a piecewise smooth closed, oriented surface with unit normal $\mathbf{n}$. If $\mathbf{F}$ is a vector field that has continuous partial derivatives on Q, then*

For our purposes, a surface is **oriented** if it has two distinct sides.

$$\iiint_Q \nabla\cdot\mathbf{F}\,dV = \iiint_Q div\,\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS \tag{2.53}$$

In (2.53), $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ is called the **outward flux** of the vector field $\mathbf{F}$ across the surface S. If S is a portion of the level curve $g(x,y) = C$ for some g, then a unit normal vector $\mathbf{n}$ may be taken to be either

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} \qquad \text{or} \qquad \mathbf{n} = -\frac{\nabla g}{\|\nabla g\|}.$$

If S is defined parametrically by

$$\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k},\ (s,t) \in R,$$

a unit normal vector to the surface is

$$\mathbf{n} = \frac{\mathbf{r}_s \times \mathbf{r}_t}{\|\mathbf{r}_s \times \mathbf{r}_t\|}$$

and (2.53) becomes

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\left(\mathbf{r}_s \times \mathbf{r}_t\right)\,dA.$$

Example 82. Find the outwardflux of the vector field

$$\mathbf{F}(x,y,z) = \left(xz + xyz^2\right)\mathbf{i} + \left(xy + x^2yz\right)\mathbf{j} + \left(yz + xy^2z\right)\mathbf{k}$$

through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

Solution. By the Divergence theorem,

$$\iint_{\text{cube surface}} \mathbf{F}\cdot\mathbf{n}\,dA = \iiint_{\text{cube interior}} \nabla\cdot\mathbf{F}\,dV.$$

Hence, without the Divergence theorem, calculating the outward flux would require six separate integrals, corresponding to the six faces of the cube. After defining $\mathbf{F}$, we compute $\nabla \cdot \mathbf{F}$ with `Div`.

`Div` is contained in the `VectorAna`… package. You do not need to reload t… `torAnalysis` package if you have loaded it during your *current* Math… session.

```
In[492]:= << Calculus`VectorAnalysis`

In[493]:= f[x_, y_, z_] = {x z + x y z^2, x y + x^2y z, y z + x y^2 z};

In[494]:= divf = Div[f[x, y, z], Cartesian[x, y, z]]
Out[494]= x + y + x y^2 + z + x^2 z + y z^2
```

The outward flux is then given by

$$\iiint_{\text{cube interior}} \nabla \cdot \mathbf{F}\, dV = \int_0^1 \int_0^1 \int_0^1 \nabla \cdot \mathbf{F}\, dz\, dy\, dx = 2,$$

which we compute with `Integrate`.

```
In[495]:= Integrate[divf, {z, 0, 1}, {y, 0, 1}, {x, 0, 1}]
Out[495]= 2
```

□

Theorem 21 (Stoke's Theorem). *Let S be an oriented surface with finite surface area, unit normal* $\mathbf{n}$, *and boundary C. Let* $\mathbf{F}$ *be a continuous vector field defined on S such that the components of* $\mathbf{F}$ *have continuous partial derivatives at each nonboundary point of S. Then,*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S curl\, \mathbf{F} \cdot \mathbf{n}\, dS. \tag{2.54}$$

In other words, the surface integral of the normal component of the curl of $\mathbf{F}$ taken over S equals the line integral of the tangential component of the field taken over C. In particular, if $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$, then

$$\int_C (P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz) = \iint_S \text{curl}\, \mathbf{F} \cdot \mathbf{n}\, dS.$$

Example 83. Verify Stoke's theorem for the vector field

$$\mathbf{F}(x,y,z) = \left(x^2 - y\right)\mathbf{i} + \left(y^2 - z\right)\mathbf{j} + \left(x + z^2\right)\mathbf{k}$$

and S the portion of the paraboloid $z = f(x,y) = 9 - \left(x^2 + y^2\right)$, $z \geq 0$.

Solution. After loading the `VectorAnalysis` package, we define $\mathbf{F}$ and f. The curl of $\mathbf{F}$ is computed with `Curl` in `curlF`.

```
In[496]:= << Calculus`VectorAnalysis`

In[497]:= capf[x_, y_, z_] = {x^2 - y, y^2 - z, x + z^2};
          f[x_, y_] = 9 - (x^2 + y^2);
```

```
In[498]:= curlcapf = Curl[capf[x, y, z], Cartesian[x, y, z]]
Out[498]= {1, -1, 1}
```

Next, we define the function $h(x, y, z) = z - f(x, y)$. A normal vector to the surface is given by ∇h. A unit normal vector, $\mathbf{n}$, is then given by $\mathbf{n} = \frac{\nabla h}{\|\nabla h\|}$, which is computed in `un`.

```
In[499]:= BoxData({h[x_, y_, z_] = z-f[x, y],  normtosurf = Grad[h[x, y, z], Cartesian[x, y, z]]})
Out[499]= -9 + x^2 + y^2 + z
Out[499]= {2 x, 2 y, 1}
```

```
In[500]:= un = Simplify[normtosurf/Sqrt[normtosurf.normtosurf]]
```

$$\text{Out[500]= } \left\{\frac{2x}{\sqrt{1+4x^2+4y^2}}, \frac{2y}{\sqrt{1+4x^2+4y^2}}, \frac{1}{\sqrt{1+4x^2+4y^2}}\right\}$$

The dot product curl $\mathbf{F} \cdot \mathbf{n}$ is computed in `g`.

```
In[501]:= g = Simplify[curlcapf.un]
```

$$\text{Out[501]= } \frac{1+2x-2y}{\sqrt{1+4x^2+4y^2}}$$

Using the surface integral evaluation formula (2.49),

In this example, R, the projection of $f(x, y)$ onto the xy-plane, is the region bounded by the graph of the circle $x^2 + y^2 = 9$.

$$\begin{aligned}\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R g\,(x, y, f(x, y))\sqrt{\left[f_x(x, y)\right]^2 + \left[f_y(x, y)\right]^2 + 1}\,dA \\ &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} g\,(x, y, f(x, y))\sqrt{\left[f_x(x, y)\right]^2 + \left[f_y(x, y)\right]^2 + 1}\,dy\,dx \\ &= 9\pi,\end{aligned}$$

which we compute with `Integrate`.

```
In[502]:= tointegrate = Simplify[(g/.z- > f[x, y])*
              Sqrt[D[f[x, y], x]^2 + D[f[x, y], y]^2 + 1]]
Out[502]= 1 + 2 x - 2 y
```

```
In[503]:= i1 = Integrate[tointegrate, {x, -3, 3},
             {y, -Sqrt[9 - x^2], Sqrt[9 - x^2]}]
Out[503]= 9 π
```

To verify Stoke's theorem, we must compute the associated line integral. Notice that the boundary of $z = f(x, y) = 9 - (x^2 + y^2)$, $z = 0$, is the circle $x^2 + y^2 = 9$ with parametrization $x = 3\cos t$, $y = 3\sin t$, $z = 0$, $0 \le t \le 2\pi$. This parametrization is substituted into $\mathbf{F}(x, y, z)$ and named `pvf`.

```
In[504]:= pvf = capf[3Cos[t], 3Sin[t], 0]
Out[504]= {9 Cos[t]^2 - 3 Sin[t], 9 Sin[t]^2, 3 Cos[t]}
```

To evaluate the line intgral along the circle, we next define the parametrization of the circle and calculate $d\mathbf{r}$. The dot product of `pvf` and `dr` represents the integrand of the line integral.

```
In[505]:= r[t_] = {3Cos[t], 3Sin[t], 0};
           dr = r′[t]
Out[505]= {-3 Sin[t], 3 Cos[t], 0}

In[506]:= tointegrate = pvf.dr;
```

As before with x and y, we instruct Mathematica to assume that t is real, compute the dot product of `pvf` and `dr` and evaluate the line integral with `Integrate`.

```
In[507]:= Integrate[tointegrate, {t, 0, 2π}]
Out[507]= 9 π
```

As expected, the result is 9π. □

2.7 Exercises

1. Compute the following by hand and check your results with Mathematica:

   (a) $\frac{d}{dx}\left(\frac{x^2}{2}+\frac{1}{3x^3}\right)$

   (b) $\frac{d}{dx}\left(xe^{-x}\right)$

   (c) $\frac{d}{dt}\left(\cos\sqrt{t}+\sqrt{\cos t}\right)$

   (d) $\frac{d}{dt}\left(1+\frac{1}{t}\right)^t$

   (e) $\frac{d}{dx}x^{\sin x}$

2. Evaluate the following antiderivatives by hand and check your results with Mathematica:

   (a) $\int\left(\frac{x^2}{2}+\frac{1}{3x^3}\right)dx$

   (b) $\int xe^{-x}\,dx$

   (c) $\int \frac{1}{t}\sin\left(\frac{1}{t^2}\right)dt$

3. Let $f(x)=x^2\sin^2(1/x)$ for $x\neq 0$ and $f(0)=0$. (a) Show that $x=0$ is a critical number of $f(x)$. (b) Explain why Theorem 3 can or cannot be used to classify $f(0)$. (c) Classify $f(0)$.

4. (a) Evaluate $\lim_{x\to 0}\frac{\sin x}{x}$. (b) Evaluate $\int\frac{\sin x}{x}\,dx$ and carefully use the Fundamental Theorem of Calculus to verify your result.

5. Determine the intervals for which $f(x)=x^{1/3}(x-4)^{1/3}(x-6)^{2/3}$ is increasing and decreasing. Generate a graph of $f(x)$ that confirms your results.

6. Classify the relative extreme values of $f(x)=2\cos x+\sin 2x$, $0\leq x\leq 2\pi$. Graph $f(x)$ on this interval.

7. Determine the intervals for which $f(x)=\sqrt[3]{x}(x+4)$ is increasing, decreasing, concave up, and concave down. Graph $f(x)$.

8. Let R be the region in the first quadrant bounded by the graphs of $y = \sin x$, $x = 0$, $x = \pi$ and the x-axis. Find the volume of the solid obtained by revolving R about (a) the x-axis, (b) the y-axis, (c) the horizontal line $y = 2$, and (d) the vertical line $x = -1$.

9. Let R be the region in the first quadrant bounded by the graphs of $y = x - x^2$ and the x-axis. Find the volume of the solid obtained by revolving R about (a) the x-axis, (b) the y-axis, (c) the line $x = 1$, and (d) the line $y = 1$. (e) Are the results the same as the results obtained in the following exercise? Why or why not?

10. Let R be the region bounded by the graphs of $y = x$ and $y = x^2$. Find the volume of the solid obtained by revolving R about (a) the x-axis, (b) the y-axis, (c) the line $x = 1$, and (d) the line $y = 1$. (e) Are the results the same as the results obtained in the previous exercise? Why or why not?

11. Determine if the following improper integrals converge or diverge.

    (a) $\int_{-\infty}^{\infty} \frac{1}{t^2+1}\,dt$
    (b) $\int_{-\infty}^{\infty} \frac{1}{x^2+3x+2}\,dx$

12. Determine if the following series converge or diverge.

    (a) $\sum_{k=1}^{\infty} \left(\frac{k}{k+1000}\right)^k$
    (b) $\sum_{k=1}^{\infty} (-1)^{k+1} k \sin\left(\frac{1}{k}\right)$
    (c) $\sum_{k=1}^{\infty} 10^{-k!}$

13. Determine the interval of convergence of each power series.

    (a) $\sum_{k=0}^{\infty} \frac{k!}{2k!} x^k$
    (b) $\sum_{k=1}^{\infty} \frac{3^k}{k} (x-1)^k$
    (c) $\sum_{k=1}^{\infty} \frac{1}{k3^k} (x+1)^k$
    (d) $\sum_{k=0}^{\infty} \left(\frac{k^2}{1+k^2}\right)^k x^k$

14. Find the Maclaurin series for $\ln\left(\frac{1+x}{1-x}\right)$. What is the interval of convergence for this series?

15. (a) Show that

$$\sum_{k=0}^{\infty} \frac{1}{2^k} \cos\left(3^k x\right)$$

converges absolutely for all values of x. (b) Use term-by-term differentiation to differentiate

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \cos\left(3^k x\right).$$

State (at least) three particular values of x that are not 2π-multiples of each other for which the resulting series diverges. (c) Provide graphical evidence that $f(x)$

is continuous everywhere but nowhere differentiable. Write a short paragraph providing justification for the graphs you provide.

16. Let
$$a_n = \frac{n!}{1000^n}.$$
(a) Compute and graph the first few terms of the sequence. (b) Based on (a), do you think the sequence converges or diverges? (c) By hand, determine if the sequence converges or diverges.

17. Show that
$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$
does not exist. *Hint:* The level curves of $f(x,y) = \frac{x^2y}{x^4+y^2}$ near $(0,0)$ look like parabolas of the form $y = ax^2$.

18. Find $f_x(0,0)$, $f_y(0,0)$, $f_{xy}(0,0)$, and $f_{yx}(0,0)$ if
$$f(x,y) = \begin{cases} \frac{xy(y^2-x^2)}{x^2+y^2}, \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0) \end{cases} .$$

19. Find the relative maximum, relative minimum, and saddle points of $f(x,y) = 6x^2y - 3x^4 - 2y^3$. Confirm your results with both three-dimensional and contour plots.

20. Find the outwardflux of the vector field
$$\mathbf{F}(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$$
through the surface S enclosed by the hemisphere $z = \sqrt{1 - x^2 - z^2}$ and the plane $z = 0$.

21. Show that the curvature of the circle with parametric equations, $x = r\cos t$, $y = r\sin t$, $0 \leq t \leq 2\pi$ is $1/r$.

22. Verify Stoke's theorem for the vector field
$$\mathbf{F}(x,y,z) = \left(y^2 - z\right)\mathbf{i} + \left(x + z^2\right)\mathbf{j} + \left(x^2 - 1\right)\mathbf{k}$$
and S the portion of the $z = f(x,y) = 25 - \left(9x^2 + 16y^2\right)$, $z \geq 0$.

23. A parametrization of the **Möbius strip** is given by $\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}$, $0 \leq s \leq \pi$, $-1 \leq t \leq 1$, where $x = (4 - t\sin s)\cos 2s$, $y = (4 - t\sin s)\sin 2s$, and $z = t\cos s$. Graph the Möbius strip.

24. Let $f(x) = \left(x\sin\left(\frac{1}{x}\right)\right)^2$ for $x \neq 0$ and $f(0) = 0$. (a) Show that $x = 0$ is a critical number. (b) Explain why the First Derivative Test can or cannot be used to classify this critical number. Use a graph to support your explanation. (c) Classify $x = 0$.

25. Let S be the surface given parametrically by $\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}$, $0 \le s \le 1/2$, $0 \le t \le 4\pi$, where $x = s\cos t - \frac{1}{2}s^2\cos 2t$, $y = -s\sin t - \frac{1}{2}s^2\sin 2t$, and $z = \frac{4}{3}s^{3/2}\cos\left(\frac{3}{2}t\right)$. (a) Graph S. (b) Explain why or why not S is orientable. *Hint:* Graph $\mathbf{n} = \frac{\mathbf{r}_s\times\mathbf{r}_t}{\|\mathbf{r}_s\times\mathbf{r}_t\|}$. (c) If S is orientable, find the area of S. *Note:* S is called **Maeder's Owl Minimal Surface**.

Chapter 3

Linear Algebra

Chapter 3 discusses Mathematica's linear algebra commands.

The examples used to illustrate the various commands are similar to examples routinely done in a one semester linear algebra course and include solving systems of linear equations and finding eigenvalues and eigenvectors of a square matrix.

3.1 Linear Systems of Equations

Given a linear system of equations, we can solve it quickly by eliminating variables in an efficient way. Given a linear system of equations, performing the operations of

1. interchanging the order of the equations,
2. multiplying an equation by a nonzero number, and
3. adding a nonzero multiple of one equation to another

result in a system equivalent to the original.

Example 84. Solve

$$x + y + z = 2 \tag{3.1}$$
$$x - 2y + 2z = 7 \tag{3.2}$$
$$x + 3y + 2z = 2 \tag{3.3}$$

Solution. We methodically eliminate. We eliminate x from (3.2) and (3.3) by adding

$-1\times$ (3.1) to (3.2) and (3.3) which results in

$$x + y + z = 2 \tag{3.4}$$
$$-3y + z = 5 \tag{3.5}$$
$$2y + z = 0 \tag{3.6}$$

Multiplying (3.5) by $-\frac{1}{3}$ gives us

$$x + y + z = 2 \tag{3.7}$$
$$y - \frac{1}{3}z = -\frac{5}{3} \tag{3.8}$$
$$2y + z = 0 \tag{3.9}$$

We now eliminate y from (3.9) by adding $-2\times$ (3.8) to (3.9) giving us

$$x + y + z = 2 \tag{3.10}$$
$$y - \frac{1}{3}z = -\frac{5}{3} \tag{3.11}$$
$$\frac{5}{3}z = \frac{1}{3} \tag{3.12}$$

Multiplying (3.12) by $\frac{3}{5}$ results in

$$x + y + z = 2 \tag{3.13}$$
$$y - \frac{1}{3}z = -\frac{5}{3} \tag{3.14}$$
$$z = 2 \tag{3.15}$$

where we see that $z = 2$. Substituting $z = 2$ into (3.14) shows us that $y = -1$. Substituting $y = -1$ and $z = 2$ into (3.13) shows us that $x = 1$. The solution to the system is $x = 1, y = -1, z = 2$.

We confirm this result with `Solve`.

```
In[508]:= Solve[{x + y + z == 2, x - 2y + 2z == 7, x + 3y + 2z == 2}]
Out[508]= {{x → 1, y → -1, z → 2}}
```

□

In the example, notice that the most difficult part is writing the variables. Also, it doesn't matter what we call the variables: solving

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 - 2x_2 + 2x_3 &= 7 \\ x_1 + 3x_2 + 2x_3 &= 2 \end{aligned}$$

is the same as solving (3.1)-(3.3).

Definition 1 (Matrix). *A **matrix** is a rectangular array.*

In a text, we usually denote matrices with bold letters.

Example 85. (a)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & -2 & 2 & 7 \\ 1 & 3 & 2 & 2 \end{pmatrix}$$

is a 3×2 matrix because it has three columns and two rows. The entries in this matrix are numbers. (b) If $z = f(x, y)$ and $z = g(x, y)$ are functions for which the first partial derivatives exist,

From a practical point of view, compute $\partial f/\partial x = f_x$ by assuming that all variables except x are constant.

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix}$$

is a 2×2 matrix because it has two rows and two columns. The entries of this matrix would usually be functions of x and y. (c) If x and y are differentiable functions of t, $x = x(t)$ and $y = y(t)$, and $\mathbf{x}$ is the 2×1 matrix $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, the matrix

$$\mathbf{x}' = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

is a 2×1 matrix because it has two rows and one column. A matrix with one column is called a **column vector**. Similarly, a matrix with one row is called a **row vector**. If the context is clear, the word row or column is omitted and a matrix with one column (or row) is referred to as a **vector**.

If $\mathbf{A}$ has n rows and m columns, we can write it as

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix} \quad \text{or} \quad \mathbf{A} = \left(a_{ij}\right),$$

where the entry in the ith row and jth column of $\mathbf{A}$ is denoted by a_{ij}, which is defined in Mathematica by entering

```
capa={{a11,a12,...,a1m},{a21,a22,...,a2m},...,{an1,an2,...,anm}}.
```

After you have defined $\mathbf{A}$ =`capa`, `capa[[i]]` returns the ith row of $\mathbf{A}$. Use `MatrixForm` to display a matrix in traditional row-and-column form.

Given the linear system with n equations and m unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nm}x_m &= b_n \end{aligned} \tag{3.16}$$

we can associate it with the matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} & b_m \end{pmatrix}.$$

Similarly, given a matrix, we can associate it with a linear system of equations.

Example 86. (a)

$$\begin{cases} x = 1 \\ y = -2 \end{cases} \iff \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

(b)

$$\begin{cases} x - y = 3 \\ 2x + y = 6 \end{cases} \iff \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 6 \end{pmatrix}$$

(c)

$$\begin{cases} 2x + 2y = 3 \\ 4x + 4y = 5 \end{cases} \iff \begin{pmatrix} 2 & 2 & 3 \\ 4 & 4 & 5 \end{pmatrix}$$

(d)

$$\begin{cases} x + z = 1 \\ y - z = 0 \end{cases} \iff \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, performing the row operations of

1. interchanging the rows of a matrix,
2. multiplying a row of a matrix by a nonzero number, and
3. adding a nonzero multiple of one row to another row of a matrix

correspond to the operations of

1. interchanging the order of the equations,
2. multiplying an equation by a nonzero number, and
3. adding a nonzero multiple of one equation to another

to the corresponding system of linear equations and vice-versa.

Example 87. Solve

$$\begin{aligned} -3x + 2y - 2z &= -10 \\ 3x - y + 2z &= 7 \\ 2x - y + z &= 6. \end{aligned}$$

Solution. The associated matrix is $\mathbf{A} = \begin{pmatrix} -3 & 2 & -2 & -10 \\ 3 & -1 & 2 & 7 \\ 2 & -1 & 1 & 6 \end{pmatrix}$, defined in `capa`, and then displayed in traditional row-and-column form with `MatrixForm`.

```
In[509]:= Clear[capa]

In[510]:= capa = {{-3, 2, -2, -10}, {3, -1, 2, 7}, {2, -1, 1, 6}};
          MatrixForm[capa]
```

Out[510]= $\begin{pmatrix} -3 & 2 & -2 & -10 \\ 3 & -1 & 2 & 7 \\ 2 & -1 & 1 & 6 \end{pmatrix}$

We eliminate methodically. First, we multiply row 1 by −1/3 so that the first entry in the first column is 1.

```
In[511]:= capa = {-1/3capa[[1]], capa[[2]], capa[[3]]}
```

Out[511]= $\left\{\left\{1, -\frac{2}{3}, \frac{2}{3}, \frac{10}{3}\right\}, \{3, -1, 2, 7\}, \{2, -1, 1, 6\}\right\}$

We now eliminate below. First, we multiply row 1 by −3 and add it to row 2 and then we multiply row 1 by −2 and add it to row 3.

```
In[512]:= capa = {capa[[1]], -3capa[[1]] + capa[[2]],
             -2capa[[1]] + capa[[3]]}
```

Out[512]= $\left\{\left\{1, -\frac{2}{3}, \frac{2}{3}, \frac{10}{3}\right\}, \{0, 1, 0, -3\}, \left\{0, \frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}\right\}\right\}$

Observe that the first nonzero entry in the second row is 1. We eliminate below this entry by adding −1/3 times row 2 to row 3.

```
In[513]:= capa = {capa[[1]], capa[[2]], -1/3 capa[[2]] + capa[[3]]}
```

Out[513]= $\left\{\left\{1, -\frac{2}{3}, \frac{2}{3}, \frac{10}{3}\right\}, \{0, 1, 0, -3\}, \left\{0, 0, -\frac{1}{3}, \frac{1}{3}\right\}\right\}$

We multiply the third row by −3 so that the first nonzero entry is 1.

```
In[514]:= capa = {capa[[1]], capa[[2]], -3capa[[3]]}

          MatrixForm[capa]
```

Out[514]= $\left\{\left\{1, -\frac{2}{3}, \frac{2}{3}, \frac{10}{3}\right\}, \{0, 1, 0, -3\}, \{0, 0, 1, -1\}\right\}$

Out[514]= $\begin{pmatrix} 1 & -\frac{2}{3} & \frac{2}{3} & \frac{10}{3} \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

This matrix is equivalent to the system

$$\begin{aligned} x - \frac{2}{3}y + \frac{2}{3}z &= \frac{10}{3} \\ y &= -3 \\ z &= -1, \end{aligned}$$

which shows us that the solution is $x = 2$, $y = -3$, $z = -1$.

Working backwards confirms this. Multiplying row 2 by 2/3 and adding to row 1 and then multiplying row 3 by −2/3 and adding to row 1 results in

```
In[515]:= capa = {2/3 capa[[2]] + capa[[1]], capa[[2]], capa[[3]]};
          capa = {-2/3 capa[[3]] + capa[[1]], capa[[2]], capa[[3]]};
          MatrixForm[capa]
```

$$\text{Out[515]=} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

which is equivalent to the system $x = 2$, $y = -3$, $z = -1$.

Equivalent results are obtained with `RowReduce`.

```
In[516]:= capa = {{-3, 2, -2, -10}, {3, -1, 2, 7}, {2, -1, 1, 6}};
          capa = RowReduce[capa]

          MatrixForm[capa]
Out[516]= {{1, 0, 0, 2}, {0, 1, 0, -3}, {0, 0, 1, -1}}
```

$$\text{Out[516]=} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Finally, we confirm the result directly with `Solve`.

```
In[517]:= Solve[{-3x + 2y - 2z == -10, 3x - y + 2z == 7, 2x - y + z == 6}]
Out[517]= {{x → 2, y → -3, z → -1}}
```

□

As illustrated in the example, `RowReduce` can be used to perform the elementary row operations on a matrix. Generally, `RowReduce[A]` reduces $\mathbf{A}$ to *reduced row echelon form*.

Example 88. Solve

$$\begin{aligned} -3x_1 + 2x_2 + 5x_3 &= -12 \\ 3x_1 - x_2 - 4x_3 &= 9 \\ 2x_1 - x_2 - 3x_3 &= 7. \end{aligned}$$

Solution. The associated matrix is $\mathbf{A} = \begin{pmatrix} -3 & 2 & 5 & -12 \\ 3 & -1 & -4 & 9 \\ 2 & -1 & -3 & 7 \end{pmatrix}$, which is reduced to row echelon form with `RowReduce`.

```
In[518]:= capa = {{-3, 2, 5, -12}, {3, -1, -4, 9}, {2, -1, -3, 7}};
          rrcapa = RowReduce[capa];
          MatrixForm[rrcapa]
Out[518]= (1  0  -1   2
           0  1   1  -3
           0  0   0   0)
```

The result shows that the original system is equivalent to

$$\begin{aligned} x_1 - x_3 &= 2 \\ x_2 + x_3 &= -3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_1 &= 2 + x_3 \\ x_2 &= -3 - x_3 \end{aligned}$$

so x_3 is *free*. That is, for any real number t, a solution to the system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2+t \\ -3-t \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

The system has infinitely many solutions.

Equivalent results are obtained with `Solve`.

```
In[519]:= Solve[{-3x1 + 2x2 + 5x3 == -12, 3x1 - x2 - 4x3 == 9,
            2x1 - x2 - 3x3 == 7}]
Solve :: "svars" :  "Equationsmaynotgivesolutions
     forallšolvev̌ariables."
Out[519]= {{x1 → 2 + x3, x2 → -3 - x3}}

In[520]:= Solve[{-3x1 + 2x2 + 5x3 == -12,
            3x1 - x2 - 4x3 == 9, 2x1 - x2 - 3x3 == 7},
           {x1, x2}]
Out[520]= {{x1 → 2 + x3, x2 → -3 - x3}}
```

□

Example 89. Solve

$$\begin{aligned} -3x_1 + 2x_2 + 5x_3 &= -14 \\ 3x_1 - x_2 - 4x_3 &= 11 \\ 2x_1 - x_2 - 3x_3 &= 8. \end{aligned}$$

Solution. The associated matrix is $\mathbf{A} = \begin{pmatrix} -3 & 2 & 5 & -14 \\ 3 & -1 & -4 & 11 \\ 2 & -1 & -3 & 8 \end{pmatrix}$, which is reduced to row echelon form with `RowReduce`.

```
In[521]:= capa = {{-3, 2, 5, -14}, {3, -1, -4, 11}, {2, -1, -3, 8}};
          RowReduce[capa]//MatrixForm
Out[521]= (1  0  -1  0
           0  1   1  0
           0  0   0  1)
```

The result shows that the original system is equivalent to

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + x_3 &= 0 \\ 0 &= 1. \end{aligned}$$

Of course, 0 is not equal to 1: the last equation is false. The system has no solutions.

We check the calculation with `Solve`. In this case, the results indicate that `Solve` cannot find any solutions to the system.

```
In[522]:= Solve[{-3x1 + 2x2 + 5x3 == -14, 3x1 - x2 - 4x3 == 11,
             2x1 - x2 - 3x3 == 8}]
Out[522]= {}
```

Generally, if Mathematica returns nothing, the result means either that there is no solution or that Mathematica cannot solve the problem. In such a situation, we must always check using another method, which we will do in Section 3.3. □

Example 90. The **nullspace** of $\mathbf{A}$ is the set of solutions to the system of equations $\mathbf{Ax} = \mathbf{0}$. Find the nullspace of $\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 & 1 & -2 \\ 3 & 3 & 1 & 2 & -1 \\ 2 & 2 & 1 & 1 & -1 \\ -1 & -1 & 0 & -1 & 0 \\ 5 & 4 & 2 & 2 & -3 \end{pmatrix}$.

Solution. Observe that row reducing $(\mathbf{A}|\mathbf{0})$ is equivalent to row reducing $\mathbf{A}$. After defining $\mathbf{A}$, we use `RowReduce` to row reduce $\mathbf{A}$.

```
In[523]:= capa = {{3, 2, 1, 1, -2}, {3, 3, 1, 2, -1},
             {2, 2, 1, 1, -1}, {-1, -1, 0, -1, 0}, {5, 4, 2, 2, -3}};
          RowReduce[capa]//MatrixForm
```

$$\text{Out[523]=} \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The result indicates that the solutions of $\mathbf{Ax} = \mathbf{0}$ are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} t \\ -s-t \\ s+t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

where s and t are any real numbers. The dimension of the nullspace, the **nullity**, is 2; a

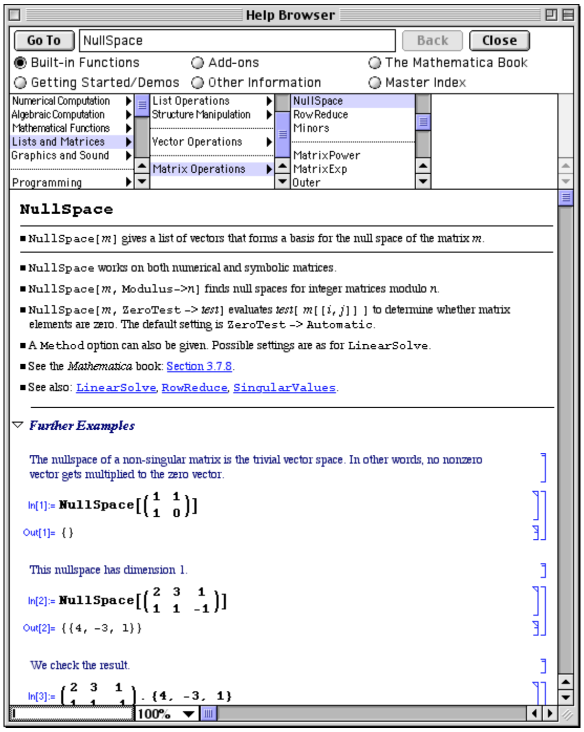


Figure 3.1: Mathematica's help for `NullSpace`

basis for the nullspace is

$$\left\{\begin{pmatrix}0\\-1\\1\\1\\0\end{pmatrix},\begin{pmatrix}1\\-1\\1\\0\\1\end{pmatrix}\right\}.$$

You can use the command `NullSpace` (see Figure 3.1 to find a basis of the nullspace directly.

```
In[524]:= NullSpace[capa]
Out[524]= {{1, -1, 1, 0, 1}, {0, -1, 1, 1, 0}}
```

□

3.2 Matrix Operations

The matrix operations of addition, subtraction, and scalar multiplication are performed

A **scalar** is a number.

in the natural way.

1. If $\mathbf{A} = \left(a_{ij}\right)$ and $\mathbf{B} = \left(b_{ij}\right)$ are $n \times m$, $\mathbf{A} \pm \mathbf{B} = \left(a_{ij} \pm b_{ij}\right)$. That is, if $\mathbf{A}$ and $\mathbf{B}$ have the same dimensions, the sum (difference) is the matrix obtained by adding (subtracting) the corresponding entries.

If $\mathbf{A}$ and $\mathbf{B}$ do not have the same dimensions, the operations of matrix addition and subtraction are not defined.

2. If c is a scalar and $\mathbf{A} = \left(a_{ij}\right)$, $c\mathbf{A} = \left(ca_{ij}\right)$. That is, to multiply a matrix by a scalar quantity, multiply each entry of the matrix by the scalar quantity.

Matrix multiplication is more complicated for the beginner. If $\mathbf{A} = \left(a_{ij}\right)$ and is $n \times k$ and $\mathbf{B} = \left(b_{ij}\right)$ is $k \times m$, $\mathbf{AB}$ is defined to be the matrix

$$\mathbf{C} = \left(c_{ij}\right) \tag{3.17}$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} = \sum_{u=1}^{k} a_{iu}b_{uj}. \tag{3.18}$$

That is, if $\mathbf{A} = \left(a_{ij}\right)$ and is $n \times k$ and $\mathbf{B} = \left(b_{ij}\right)$ is $k \times m$, $\mathbf{AB}$ is the matrix $\mathbf{C} = \left(c_{ij}\right)$ where c_{ij} is obtained by multiplying each entry in the ith row of $\mathbf{A}$ by the corresponding entry in the jth column of $\mathbf{B}$ and adding the result. Note that if $\mathbf{A}$ and $\mathbf{B}$ do not have the appropriate dimensions, the matrix product is *not* defined.

If the matrix product is defined, enter `A.B` to compute the product $\mathbf{AB}$.

Example 91. Let $\mathbf{A} = \begin{pmatrix} -8 & -5 & -3 \\ -3 & 9 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 7 & 5 \\ 4 & 6 \\ 5 & -5 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 4 & -8 & -9 \\ 9 & 4 & -6 \\ 7 & 6 & 5 \end{pmatrix}$, and $\mathbf{D} = \begin{pmatrix} -5 & 9 \\ -6 & -5 \end{pmatrix}$. If defined, perform each computation: (a) $\mathbf{AB}$, (b) $\mathbf{BAC}$, (c) $\mathbf{AB} + \mathbf{D}$, (d) $\mathbf{AC}$, and (e) $\mathbf{BC}$.

Solution. After defining $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$,

```
In[525]:= capa = {{-8, -5, -3}, {-3, 9, 5}};
          capb = {{7, 5}, {4, 6}, {5, -5}};
          capc = {{4, -8, -9}, {9, 4, -6}, {7, 6, 5}};
          capd = {{-5, 9}, {-6, -5}};
```

we perform each defined calculation. Entering

```
In[526]:= capa.capb

          MatrixForm[%]
Out[526]= {{-91, -55}, {40, 14}}
Out[526]= ( -91  -55 )
          ( 40    14 )
```

computes $\mathbf{AB}$. Entering

```
In[527]:= capb.capa.capc

          MatrixForm[%]
Out[527]= {{-166, 632, 599}, {232, 644, 336}, {-1010, -320, 445}}
Out[527]= ( -166   632   599 )
          (  232   644   336 )
          ( -1010  -320  445 )
```

computes **BAC**. Entering

```
In[528]:= capa.capb + capd

          MatrixForm[%]
Out[528]= {{-96, -46}, {34, 9}}
Out[528]= ( -96  -46 )
          (  34    9 )
```

computes **AB** + **D**. Entering

```
In[529]:= capa.capc

          MatrixForm[%]
Out[529]= {{-98, 26, 87}, {104, 90, -2}}
Out[529]= ( -98  26  87 )
          ( 104  90  -2 )
```

computes **AC**. Entering

```
capb . capc

Dot::dotsh : Tensors {{7, 5}, {4, 6}, {5, -5}} and
   {{4, -8, -9}, {9, 4, -6}, {7, 6, 5}} have incompatible
   shapes.

{{7, 5}, {4, 6}, {5, -5}}.{{4, -8, -9}, {9, 4, -6}, {7, 6, 5}}
```

returns an error message because **BC** is not defined. □

Example 92. The $n \times n$ **identity matrix** is the matrix **I** with 1's down the diagonal and 0's elsewhere. If **A** is $n \times n$, $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$. We verify this equation if **A** is 2×2.

```
In[530]:= id2 = {{1, 0}, {0, 1}};
          {{a, b}, {c, c}}.id2//MatrixForm

          id2.{{a, b}, {c, d}}//MatrixForm
Out[530]= ( a  b )
          ( c  c )
Out[530]= ( a  b )
          ( c  d )
```

3.3 Determinants

Let $\mathbf{A} = \left(a_{ij}\right)$ be an $n \times n$ matrix. The **cofactor** of a_{ij} is $(-1)^{i+j}\mathbf{C}_{ij}$, where $\mathbf{C}_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of **A**. The **determinant** of **A** is

Definition 2 (Determinant). *The **determinant** of the* 1×1 *matrix* $\mathbf{A} = (a)$ *is* a. *The **determinant** of the* 2×2 *matrix* $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is*

$$|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For $n > 2$, *the **determinant** of the* $n \times n$ *matrix* $\mathbf{A}$ *is defined inductively by*

$$|\mathbf{A}| = \sum_i (-1)^{i+j} a_{ij} \left|\mathbf{C}_{ij}\right|. \tag{3.19}$$

If $\mathbf{A}$ is a square matrix, the command `Det[A]` computes the determinant of $\mathbf{A}$.

Example 93. Calculate the determinant of (a) $\mathbf{A} = \begin{pmatrix} -6 & -2 & -5 \\ -9 & 0 & -1 \\ 9 & 0 & -8 \end{pmatrix}$ and (b) $\mathbf{B} = \begin{pmatrix} 2 & -8 & 1 & -5 \\ 8 & -5 & 8 & 4 \\ 7 & 1 & 7 & -9 \\ -8 & -8 & 3 & 8 \end{pmatrix}$.

Solution. (a) To compute $|\mathbf{A}|$ by hand, we choose to expand along the second column:

$$\begin{vmatrix} -6 & -2 & -5 \\ -9 & 0 & -1 \\ 9 & 0 & -8 \end{vmatrix} = (-1)^{1+2} \cdot -2 \cdot \begin{vmatrix} -9 & -1 \\ 9 & -8 \end{vmatrix} = 2 \cdot (72 - -9) = 162.$$

We check the calculation with `Det`. For (b), we use `Det`.

```
In[531]:= Det[{{-6, -2, -5}, {-9, 0, -1}, {9, 0, -8}}]
Out[531]= 162

In[532]:= Det[{{2, -8, 1, -5}, {8, -5, 8, 4}, {7, 1, 7, -9},
             {-8, -8, 3, 8}}]
Out[532]= 11047
```

□

3.3.1 Inverses

Matrices $\mathbf{A}$ and $\mathbf{B}$ are inverses if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. If $\mathbf{A}$ has an inverse, we denote the inverse by $\mathbf{A}^{-1}$. The square matrix $\mathbf{A}$ has an inverse if and only if $|\mathbf{A}| \neq 0$.

If $|\mathbf{A}| \neq 0$, the inverse of $\mathbf{A}$ can be computed using the formula

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \mathbf{A}^a, \tag{3.20}$$

where $\mathbf{A}^a$ is the *transpose of the cofactor matrix*.

The **cofactor matrix**, $\mathbf{A}^c$, of $\mathbf{A}$ is the matrix obtained by replacing each element of $\mathbf{A}$ by its cofactor.

If $\mathbf{A}$ has an inverse, reducing the matrix $(\mathbf{A}|\mathbf{I})$ to reduced row echelon form results in $(\mathbf{I}|\mathbf{A}^{-1})$. This method is often easier to implement than (3.20).

It is particularly useful to memorize the inverse formula for a 2×2 matrix: if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $|\mathbf{A}| \neq 0$,

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

```
In[533]:= RowReduce[{{a, b, 1, 0}, {c, d, 0, 1}}]
```

$$Out[533]= \left\{\left\{1, 0, \frac{d}{-b\,c + a\,d}, \frac{b}{b\,c - a\,d}\right\}, \left\{0, 1, -\frac{c}{-b\,c + a\,d}, \frac{a}{-b\,c + a\,d}\right\}\right\}$$

If $\mathbf{A}$ has an inverse, the command `Inverse[A]` computes $\mathbf{A}^{-1}$.

```
In[534]:= Inverse[{{a, b}, {c, d}}]
```

$$Out[534]= \left\{\left\{\frac{d}{-b\,c + a\,d}, -\frac{b}{-b\,c + a\,d}\right\}, \left\{-\frac{c}{-b\,c + a\,d}, \frac{a}{-b\,c + a\,d}\right\}\right\}$$

Example 94. Calculate $\mathbf{A}^{-1}$ if $\mathbf{A} = \begin{pmatrix} -2 & 2 & 1 \\ 0 & -2 & 2 \\ -2 & -1 & -1 \end{pmatrix}$.

Solution. After defining $\mathbf{A}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we compute $|\mathbf{A}| = 12$, so $\mathbf{A}^{-1}$ exists.

```
In[535]:= << LinearAlgebra‘MatrixManipulation‘;
          capa = {{2, -2, 1}, {0, -2, 2}, {-2, -1, -1}};
          i3 = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}};

In[536]:= Det[capa]
Out[536]= 12
```

We use `AppendRows` (see Figure 3.2 to form the matrix $(\mathbf{A}|\mathbf{I})$

`AppendRows` is contained in the `Matrix-Manipulation` package that is located in the **LinearAlgebra** directory.

```
In[537]:= ai3 = AppendRows[capa, i3];
          MatrixForm[ai3]
```

$$Out[537]= \begin{pmatrix} 2 & -2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 & 1 & 0 \\ -2 & -1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

and then use `RowReduce` to reduce $(\mathbf{A}|\mathbf{I})$ to row echelon form.

```
In[538]:= RowReduce[ai3]
```

$$Out[538]= \left\{\left\{1, 0, 0, \frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}\right\}, \left\{0, 1, 0, -\frac{1}{3}, 0, -\frac{1}{3}\right\}, \left\{0, 0, 1, -\frac{1}{3}, \frac{1}{2}, -\frac{1}{3}\right\}\right\}$$

The result indicates that $\mathbf{A}^{-1} = \begin{pmatrix} 1/3 & -1/4 & -1/6 \\ -1/3 & 0 & -1/3 \\ -1/3 & 1/2 & -1/3 \end{pmatrix}$. We check this result with `Inverse`.

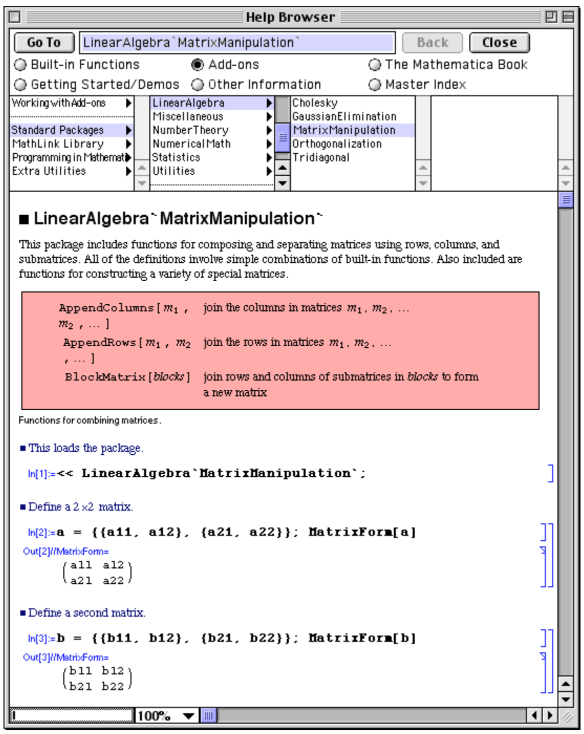


Figure 3.2: The `MatrixManipulation` package

```
In[539]:= Inverse[capa]
Out[539]= {{1/3, -1/4, -1/6}, {-1/3, 0, -1/3}, {-1/3, 1/2, -1/3}}
```

□

3.3.2 Linear Systems of Equations

Consider the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n. \end{aligned} \tag{3.21}$$

In matrix form, the system is written as $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

If $\mathbf{A}$ is invertible, it follows that $\mathbf{Ax} = \mathbf{b}$ has solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

Cramer's Rule

An alternative method of solving $\mathbf{Ax} = \mathbf{b}$ is given by *Cramer's rule*.

Let $\mathbf{A}_i$ denote the matrix obtained by replacing the *i*th column of $\mathbf{A}$ by $\mathbf{b}$. Then,

$$x_i = \frac{|\mathbf{A}_i|}{|\mathbf{A}|}.$$

The command `LinearSolve[A,b]` (see Figure 3.3) solves $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$.

Example 95. Solve each system: (a)

$$\begin{aligned} 2x_1 - 2x_2 - 2x_3 + x_4 &= 1 \\ -x_1 - 2x_2 + 2x_3 + x_4 &= -2 \\ 2x_1 - x_2 + 2x_3 &= 1 \\ -2x_1 + 2x_2 - x_3 - x_4 &= 2 \end{aligned}$$

(b)

$$\begin{aligned} -7x_1 + 4x_2 + 15x_3 + 18x_4 &= -7 \\ -x_1 + x_2 + 3x_3 + 3x_4 &= -1 \\ 5x_1 - 2x_2 - 9x_3 - 12x_4 &= 5 \\ -4x_1 + 2x_2 + 8x_3 + 10x_4 &= -4. \end{aligned}$$

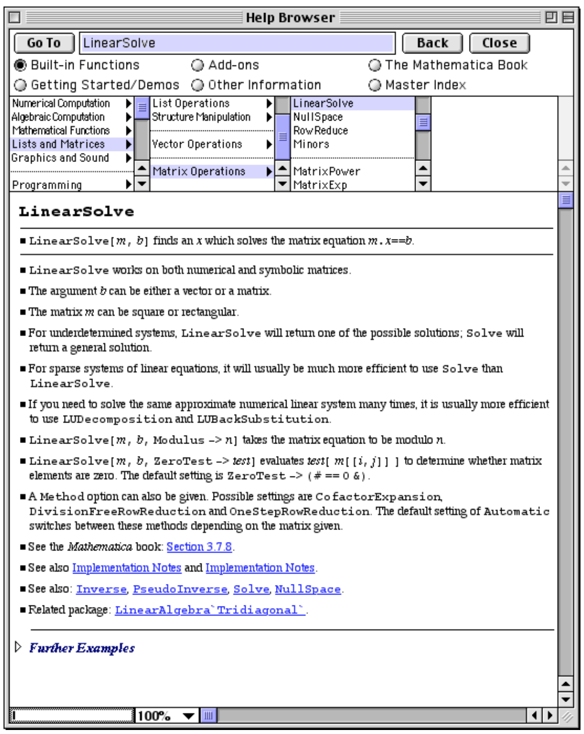


Figure 3.3: Mathematica's help for `LinearSolve`

Solution. (a) In matrix form, the system is equivalent to $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} 2 & -2 & -2 & 1 \\ -1 & -2 & 2 & 1 \\ 2 & -1 & 2 & 0 \\ -2 & 2 & -1 & -1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 2 \end{pmatrix}$. After defining $\mathbf{A}$ and $\mathbf{b}$, we see that $\mathbf{A}^{-1}$ exists by computing $|\mathbf{A}| = 9$ with `Det`.

```
In[540]:= << LinearAlgebra‘MatrixManipulation‘;
          capa = {{2, -2, -2, 1},
             {-1, -2, 2, 1}, {2, -1, 2, 0}, {-2, 2, -1, -1}};
          b = {1, -2, 1, 2};

In[541]:= Det[capa]
Out[541]= 9
```

The inverse is then found with `Inverse`.

```
In[542]:= ai = Inverse[capa]
```

$$\text{Out[542]=}\ \left\{\left\{-\frac{1}{9}, -\frac{1}{3}, 0, -\frac{4}{9}\right\}, \left\{-\frac{8}{9}, -\frac{2}{3}, -1, -\frac{14}{9}\right\}, \left\{-\frac{1}{3}, 0, 0, -\frac{1}{3}\right\}, \left\{-\frac{11}{9}, -\frac{2}{3}, -2, -\frac{26}{9}\right\}\right\}$$

We obtain the solution by computing $\mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} -1/3 \\ -11/3 \\ -1 \\ -23/3 \end{pmatrix}$. The result indicates that $x_1 = -1/3$, $x_2 = -11/3$, $x_3 = -1$, and $x_4 = -23/3$.

```
In[543]:= ai.b
```

$$\text{Out[543]=}\ \left\{-\frac{1}{3}, -\frac{11}{3}, -1, -\frac{23}{3}\right\}$$

We check the result with `LinearSolve`.

```
In[544]:= LinearSolve[capa, b]
```

$$\text{Out[544]=}\ \left\{-\frac{1}{3}, -\frac{11}{3}, -1, -\frac{23}{3}\right\}$$

Alternatively, we check the result by reducing the augmented matrix $(\mathbf{A}|\mathbf{b})$ to row echelon form with `AppendRows` and `RowReduce`.

```
In[545]:= b = Map[{#}&, b]
Out[545]= {{1}, {-2}, {1}, {2}}

In[546]:= augb = AppendRows[capa, b]
Out[546]= {{2, -2, -2, 1, 1}, {-1, -2, 2, 1, -2}, {2, -1, 2, 0, 1},
            {-2, 2, -1, -1, 2}}

In[547]:= RowReduce[augb]//MatrixForm
```

Out[547]=
$$\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & -\frac{11}{3} \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -\frac{23}{3} \end{pmatrix}$$

(b) In matrix form, the system is equivalent to $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} -7 & 4 & 15 & 18 \\ -1 & 1 & 3 & 3 \\ 5 & -2 & -9 & -12 \\ -4 & 2 & 8 & 10 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} -7 \\ -1 \\ 5 \\ -4 \end{pmatrix}$. After definining $\mathbf{A}$ and $\mathbf{b}$, we use `det` to see that $|\mathbf{A}| = 0$–$\mathbf{A}^{-1}$ does not exist. Consequently, the system may have no solutions or infinitely many solutions.

```
In[548]:= capa = {{-7, 4, 15, 18},
              {-1, 1, 3, 3}, {5, -2, -9, -12}, {-4, 2, 8, 10}};
          b = {-7, -1, 5, -4};
In[549]:= Det[capa]
Out[549]= 0
```

We use `AppendRows` to form the augmented matrix $(\mathbf{A}|\mathbf{b})$,

```
In[550]:= bvec = Map[{#}&, b]
Out[550]= {{-7}, {-1}, {5}, {-4}}
In[551]:= augb = AppendRows[capa, bvec];
          MatrixForm[augb]
```

Out[551]=
$$\begin{pmatrix} -7 & 4 & 15 & 18 & -7 \\ -1 & 1 & 3 & 3 & -1 \\ 5 & -2 & -9 & -12 & 5 \\ -4 & 2 & 8 & 10 & -4 \end{pmatrix}$$

which we then reduce using `RowReduce`. The result indicates that the system has infinitely many solutions.

```
In[552]:= RowReduce[augb]//MatrixForm
```

Out[552]=
$$\begin{pmatrix} 1 & 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By hand, we write the solutions as follows. Let s and t denote real numbers. Then, any solution to the system takes the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

We check the result with `LinearSolve`. Note that `LinearSolve` only returns one solution.

```
In[553]:= LinearSolve[capa, b]
Out[553]= {1, 0, 0, 0}
```

However, when we use `Solve`, we obtain infinitely many solutions.

```
In[554]:= capa.{x[1], x[2], x[3], x[4]} == b
Out[554]= {-7 x[1] + 4 x[2] + 15 x[3] + 18 x[4],
            -x[1] + x[2] + 3 x[3] + 3 x[4], 5 x[1] - 2 x[2] - 9 x[3] - 12 x[4],
            -4 x[1] + 2 x[2] + 8 x[3] + 10 x[4]} ==
          {-7, -1, 5, -4}
In[555]:= Solve[capa.{x[1], x[2], x[3], x[4]} == b]
Solve :: "svars" : "Equationsmaynotgivesolutions
     forallšolvev̌ariables."
Out[555]= {{x[1] → 1 + x[3] + 2 x[4], x[2] → -2 x[3] - x[4]}}
```

□

3.4 Eigenvalues and Eigenvectors

Let $\mathbf{A}$ be an $n \times n$ matrix. λ is an **eigenvalue** of $\mathbf{A}$ if there is a *nonzero* vector, $\mathbf{v}$, called an **eigenvector**, satisfying

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}. \tag{3.22}$$

We find the eigenvalues of $\mathbf{A}$ by solving the **characteristic polynomial**

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{3.23}$$

for λ. Once we find the eigenvalues, the corresponding eigenvectors are found by solving

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0} \tag{3.24}$$

for $\mathbf{v}$.

If $\mathbf{A}$ is a square matrix,

```
Eigenvalues[A]
```

finds the eigenvalues of $\mathbf{A}$,

```
Eigenvectors[A]
```

finds the eigenvectors, and

```
Eigensystem[A]
```

finds the eigenvalues and corresponding eigenvectors. `CharacteristicPolynomial[A,lambda]` finds the characteristic polynomial of $\mathbf{A}$.

Example 96. Find the eigenvalues and corresponding eigenvectors for each of the following matrices. (a) $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ (d) $\mathbf{A} = \begin{pmatrix} -1/4 & 2 \\ -8 & -1/4 \end{pmatrix}$

Solution. (a) We begin by finding the eigenvalues. Solving

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} -3-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 5 = 0$$

gives us $\lambda_1 = -5$ and $\lambda_2 = -1$.

Observe that the same results are obtained using `CharacteristicPolynomial` and `Eigenvalues`.

```
In[556]:= capa = {{-3, 2}, {2, -3}};
          CharacteristicPolynomial[capa, λ]//Factor

          e1 = Eigenvalues[capa]
Out[556]= (1 + λ) (5 + λ)
Out[556]= {-5, -1}
```

We now find the corresponding eigenvectors. Let $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ be an eigenvector corresponding to λ_1, then

$$\begin{aligned} \left(\mathbf{A} - \lambda_1 \mathbf{I}\right) \mathbf{v}_1 &= \mathbf{0} \\ \left[\begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix} - (-5)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

which row reduces to

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

That is, $x_1 + y_1 = 0$ or $x_1 = -y_1$. Hence, for any value of $y_1 \neq 0$,

$$\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -y_1 \\ y_1 \end{pmatrix} = y_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to λ_1. Of course, this represents infinitely many vectors. But, they are all linearly dependent. Choosing $y_1 = 1$ yields $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Note that you might have chosen $y_1 = -1$ and obtained $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. However, both of our results are "correct" because these vectors are linearly dependent.

Similarly, letting $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be an eigenvector corresponding to λ_2 we solve $\left(\mathbf{A} - \lambda_2 \mathbf{I}\right)\mathbf{v}_1 = \mathbf{0}$:

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, $x_2 - y_2 = 0$ or $x_2 = y_2$. Hence, for any value of $y_2 \neq 0$,

$$\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_2 \end{pmatrix} = y_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to λ_2. Choosing $y_2 = 1$ yields $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We confirm these results using `RowReduce`.

```
In[557]:= i2 = {{1, 0}, {0, 1}};
          ev1 = capa - e1[[1]] i2
Out[557]= {{2, 2}, {2, 2}}

In[558]:= RowReduce[ev1]
Out[558]= {{1, 1}, {0, 0}}

In[559]:= ev2 = capa - e1[[2]] i2

          RowReduce[ev2]
Out[559]= {{-2, 2}, {2, -2}}
Out[559]= {{1, -1}, {0, 0}}
```

We obtain the same results using `Eigenvectors` and `Eigensystem`.

```
In[560]:= Eigenvectors[capa]

          Eigensystem[capa]
Out[560]= {{-1, 1}, {1, 1}}
Out[560]= {{-5, -1}, {{-1, 1}, {1, 1}}}
```

(b) In this case, we see that $\lambda = 2$ has multiplicity 2. There is only one linearly independent eigenvector, $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, corresponding to λ.

```
In[561]:= capa = {{1, -1}, {1, 3}};
          Factor[CharacteristicPolynomial[capa, λ]]

          Eigenvectors[capa]

          Eigensystem[capa]
```

```
Out[561]= (-2 + λ)^2
Out[561]= {{-1, 1}, {0, 0}}
Out[561]= {{2, 2}, {{-1, 1}, {0, 0}}}
```

(c) The eigenvalue $\lambda_1 = 2$ has corresponding eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The eigenvalue $\lambda_{2,3} = -1$ has multiplicity 2. In this case, there are two linearly independent eigenvectors corresponding to this eigenvalue: $\mathbf{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{v_3} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

```
In[562]:= capa = {{0, 1, 1}, {1, 0, 1}, {1, 1, 0}};
          Factor[CharacteristicPolynomial[capa, λ]]

          Eigenvectors[capa]

          Eigensystem[capa]
Out[562]= -(-2 + λ) (1 + λ)^2
Out[562]= {{-1, 0, 1}, {-1, 1, 0}, {1, 1, 1}}
Out[562]= {{-1, -1, 2}, {{-1, 0, 1}, {-1, 1, 0}, {1, 1, 1}}}
```

(d) In this case, the eigenvalues $\lambda_{1,2} = -\frac{1}{4} \pm 4i$ are complex conjugates. We see that the eigenvectors $\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 2 \end{pmatrix} i$ are complex conjugates as well.

```
In[563]:= capa = {{-1/4, 2}, {-8, -1/4}};
          Eigenvectors[capa]

          Eigensystem[capa]
Out[563]= {{i, 2}, {-i, 2}}
Out[563]= {{-1/4 - 4 i, -1/4 + 4 i}, {{i, 2}, {-i, 2}}}
```

□

3.5 Exercises

1. Let $\mathbf{A} = \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 5 \\ -5 & 2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -3 & 3 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} -4 & 3 \\ -3 & 5 \\ 2 & 4 \end{pmatrix}$, and $\mathbf{E} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$. If defined, calculate each of the following by hand and confirm your result with Mathematica. If the operation is not defined, state why.

   (a) $|\mathbf{A}|$

   (b) $\mathbf{A} - \mathbf{B}$

(c) $\mathbf{BA}$

(d) $\mathbf{A}^{-1}$

(e) $|\mathbf{E}|$

(f) $\mathbf{E}^{-1}$

(g) $\mathbf{AD}$

(h) $\mathbf{DA}$

(i) $(\mathbf{CD}) + \mathbf{B}$

(j) $(\mathbf{AE}) - \mathbf{C}$

(k) $(\mathbf{CD})\,\mathbf{E}$

(l) $(\mathbf{DC})\,\mathbf{E}$

2. Calculate:

(a) $\begin{vmatrix} 7 & 6 \\ -2 & -9 \end{vmatrix}$

(b) $\begin{vmatrix} -5 & -1 & -4 \\ 6 & -7 & -8 \\ 7 & 4 & -6 \end{vmatrix}$

(c) $\begin{vmatrix} -5 & -6 & -2 & -3 \\ 3 & -3 & -4 & 8 \\ -7-6 & 2 & 2 & \\ -6 & -6 & 0 & 2 \end{vmatrix}$

3. Find the eigenvalues and corresponding eigenvectors for each matrix.

(a) $\mathbf{A} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix}$

(d) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

(e) $\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4. Solve each of the following linear systems using at least two different methods.

(a)

$$\begin{aligned}
-14x_1 - 2x_2 + x_3 - x_4 &= 1 \\
-8x_1 - x_2 + x_3 - x_4 &= -1 \\
8x_1 - 2x_2 + x_3 &= -8 \\
16x_1 + 2x_2 - x_3 + x_4 &= -2
\end{aligned}$$

(b)

$$\begin{aligned}
-14x_1 - 2x_2 + x_3 + 18x_4 &= 3 \\
-8x_1 - x_2 + x_3 + 10x_4 &= 1 \\
8x_1 - 2x_2 + x_3 - 4x_4 &= -8 \\
16x_1 + 2x_2 - x_3 - 20x_4 &= -4
\end{aligned}$$

(c)

$$\begin{aligned}
-14x_1 - 2x_2 + 12x_3 - 12x_4 &= 5 \\
-8x_1 - x_2 + 7x_3 - 7x_4 &= 3 \\
8x_1 - 2x_2 - 10x_3 + 10x_4 &= -6 \\
16x_1 + 2x_2 - 14x_3 + 14x_4 &= -6
\end{aligned}$$

(d)

$$\begin{aligned}
-14x_1 - 2x_2 + 12x_3 - 12x_4 &= 4 \\
-8x_1 - x_2 + 7x_3 - 7x_4 &= 2 \\
8x_1 - 2x_2 - 10x_3 + 10x_4 &= -6 \\
16x_1 + 2x_2 - 14x_3 + 14x_4 &= -5
\end{aligned}$$

Chapter 4

Differential Equations

Chapter 4 discusses Mathematica's differential equations commands. The examples used to illustrate the various commands are similar to examples routinely done in a one semester differential equations course.

4.1 First-Order Differential Equations

4.1.1 Separable Equations

Because they are solved by integrating, separable differential equations are usually the first introduced in the introductory differential equations course.

Definition 3 (Separable Differential Equation). *A differential equation of the form*

$$f(y)\,dy = g(x)\,dx \tag{4.1}$$

*is called a first-order **separable differential equation**.*

We solve separable differential equations by integrating.

Remark. The command

```
DSolve[y'[t]==f[t,y[t]],y[t],t]
```

attempts to solve $y' = dy/dt = f(t,y)$ for y. (See Figure 4.1.)

Example 97. Solve each of the following equations. (a) $y' - y^2 \sin t = 0$ (b) $y' = \alpha y\left(1 - \frac{1}{K}y\right)$, K, $\alpha > 0$ constant.

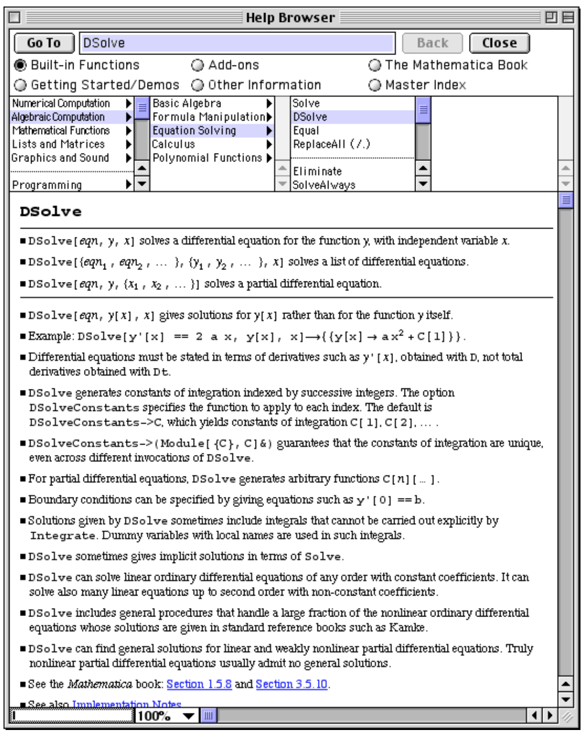


Figure 4.1: Mathematica's `DSolve` help window

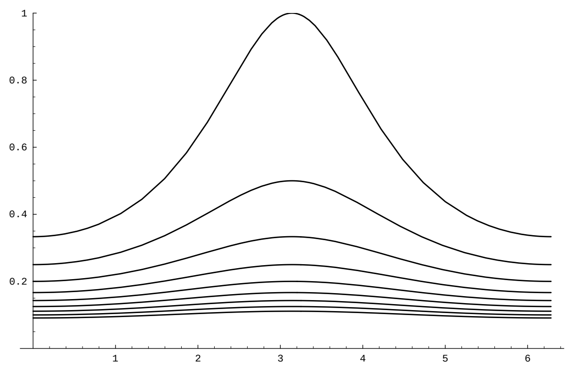


Figure 4.2: Several solutions of $y' - y^2 \sin t = 0$

Solution. (a) The equation is separable:

$$\begin{aligned} \frac{1}{y^2} dy &= \sin t \, dt \\ \int \frac{1}{y^2} dy &= \int \sin t \, dt \\ -\frac{1}{y} &= -\cos t + C \\ y &= \frac{1}{\cos t + C}. \end{aligned}$$

We check our result with `DSolve`.

```
In[564]:= sola = DSolve[y'[t] - y[t]^2 Sin[t] == 0, y[t], t]
Out[564]= {{y[t] -> 1/(-C[1] + Cos[t])}}
```

Observe that the result is given as a list. The formula for the solution is the second part of the first part of the first part of `sola`.

```
In[565]:= sola[[1, 1, 2]]
Out[565]= 1/(-C[1] + Cos[t])
```

We then graph the solution for various values of C with `Plot` in Figure 4.2.

To graph the list of functions `list` for $a \leq x \leq b$, enter `Plot[Evaluate[list],{x,a,b}]`.

```
In[566]:= toplota = Table[sola[[1, 1, 2]]/.C[1]- > -i, {i, 2, 10}]
Out[566]= {1/(2 + Cos[t]), 1/(3 + Cos[t]), 1/(4 + Cos[t]), 1/(5 + Cos[t]),
           1/(6 + Cos[t]), 1/(7 + Cos[t]), 1/(8 + Cos[t]), 1/(9 + Cos[t]),
           1/(10 + Cos[t])}
```

```
In[567]:= Plot[Evaluate[toplota], {t, 0, 2π}, PlotRange- > {0, 1},
            AxesOrigin- > {0, 0}]
```

`expression /. x->y` replaces all occurrences of x in *expression* by y. `Table[a[k],{k,n,m}]` generates the list $a_n, a_{n+1}, \ldots, a_{m-1}, a_m$.

(b) After separating variables, we use partial fractions to integrate.

$$\begin{aligned} y' &= \alpha y\left(1 - \frac{1}{K}y\right) \\ \frac{1}{\alpha y\left(1-\frac{1}{K}y\right)}dy &= dt \\ \frac{1}{\alpha}\left(\frac{1}{y}+\frac{1}{K-y}\right) &= dt \\ \frac{1}{\alpha}\left(\ln|y| - \ln|K-y|\right) &= C_1 t \\ \frac{y}{K-y} &= Ce^{\alpha t} \\ y &= \frac{CKe^{\alpha t}}{Ce^{\alpha t}-1} \end{aligned}$$

We check the calculations with Mathematica. First, we use `Apart` to find the partial fraction decomposition of $\frac{1}{\alpha y\left(1-\frac{1}{K}y\right)}$.

```
In[568]:= s1 = Apart[1/(α y (1 - 1/k y)), y]
Out[568]= 1/(y α) - 1/((-k + y) α)
```

Then, we use `Integrate` to check the integration.

```
In[569]:= s2 = Integrate[s1, y]
Out[569]= Log[y]/α - Log[-k + y]/α
```

Last, we use use `Solve` to solve $\frac{1}{\alpha}(\ln|y| - \ln|K-y|) = ct$ for y.

```
In[570]:= Solve[s2 == c t, y]
Out[570]= {{y → (e^(c t α) k)/(-1 + e^(c t α))}}
```

We can use `DSolve` to find a general solution of the equation

```
In[571]:= solb = DSolve[y'[t] == α y[t] (1 - 1/k y[t]), y[t], t]
Out[571]= {{y[t] → (e^(t α) k)/(e^(t α) - e^C[1])}}
```

as well as find the solution that satisfies the initial condition $y(0) = y_0$.

```
In[572]:= solc = DSolve[{y'[t] ==  y[t] (1 -  y[t]), y[0] == y0}, y[t], t]
Out[572]= {{y[t] → (e^t y0)/(1 - y0 + e^t y0)}}
```

The equation $y' = \alpha y\left(1 - \frac{1}{K}y\right)$ is called the **Logistic equation** (or **Verhulst equation**) and is used to model the size of a population that is not allowed to grow in an unbounded manner. Assuming that $y(0) > 0$, then all solutions of the equation have the property that $\lim_{t\to\infty} = K$.

See Figure 4.20.

To see this, we set $\alpha = K = 1$ and use `PlotVectorField`, which is contained in the `PlotField` package that is located in the **Graphics** directory to graph the direction field associated with the equation in Figure 4.3.

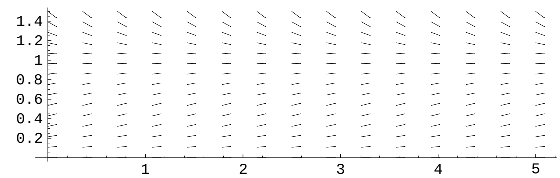


Figure 4.3: A typical direction field for the Logistic equation

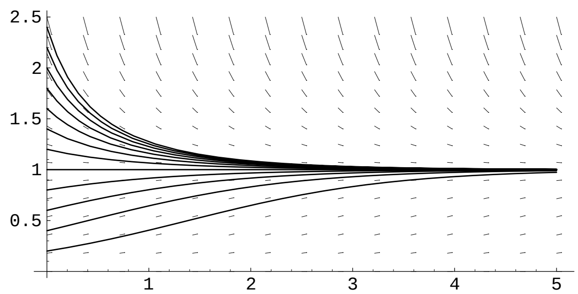


Figure 4.4: A typical direction field for the Logistic equation along with several solutions

```
In[573]:= << Graphics`PlotField`;
          pvf = PlotVectorField[{1, y(1 - y)}, {t, 0, 5},
              {y, 0, 5/2}, HeadLength- > 0, Axes- > Automatic]
```

The property is more easily seen when we graph various solutions along with the direction field as done next in Figure 4.4.

```
In[574]:= toplot = Table[solc[[1, 1, 2]]/.y0- > i/5, {i, 1, 12}];
          sols = Plot[Evaluate[toplot],
              {t, 0, 5}, DisplayFunction- > Identity];
          Show[pvf, sols]
```

□

4.1.2 Linear Equations

Definition 4 (First-Order Linear Equation). *A differential equation of the form*

$$a_1(t)\frac{dy}{dt} + a_0(t)y = f(t), \tag{4.2}$$

*where $a_1(t)$ is not identically the zero function, is a first-order **linear differential equation**.*

Assuming that $a_1(t)$ is not identically the zero function, dividing (4.2) by $a_1(t)$ gives us the **standard form** of the first-order linear equation:

$$\frac{dy}{dt} + p(t)y = q(t). \tag{4.3}$$

If $q(t)$ is identically the zero function, we say that the equation is **homogeneous**. The **corresponding homogeneous equation** of (4.3) is

$$\frac{dy}{dt} + p(t)y = 0. \tag{4.4}$$

Observe that (4.4) is separable:

$$\begin{aligned}\frac{dy}{dt} + p(t)y &= 0\\ \frac{1}{y}dy &= -p(t)\,dt\\ \ln|y| &= -\int p(t)\,dt + C\\ y &= Ce^{-\int p(t)\,dt}.\end{aligned}$$

A **particular solution** is a specific solution to the equation that does not contain any arbitrary constants.

Notice that any constant multiple of a solution to a linear homogeneous equation is also a solution. Now suppose that y is any solution of (4.3) and y_p is a particular solution of (4.3). Then,

$$\begin{aligned}\left(y - y_p\right)' + p(t)\left(y - y_p\right) &= y' + p(t)y - \left(y_p{}' + p(t)y_p\right)\\ &= q(t) - q(t) = 0.\end{aligned}$$

Thus, $y - y_p$ is a solution to the corresponding homogeneous equations of (4.3). Hence,

$$\begin{aligned}y - y_p &= Ce^{-\int p(t)\,dt}\\ y &= Ce^{-\int p(t)\,dt} + y_p\\ y &= y_h + y_p,\end{aligned}$$

where $y_h = Ce^{-\int p(t)\,dt}$. That is, a general solution of (4.3) is

$$y = y_h + y_p,$$

where y_p is a particular solution to the nonhomogeneous equation and y_h is a general solution to the corresponding homogeneous equation. Thus, to solve (4.3), we need to first find a general solution to the corresponding homogeneous equation, y_h, which we can accomplish through separation of variables, and then find a particular solution, y_p, to the nonhomogeneous equation.

If y_h is a solution to the corresponding homogeneous equation of (4.3) then for any constant C, Cy_h is also a solution to the corresponding homogeneous equation. Hence, it is impossible to find a particular solution to (4.3) of this form. Instead, we search for a particular solution of the form $y_p = u(t)y_h$, where $u(t)$ is *not* a constant function.

Assuming that a particular solution, y_p, to (4.3) has the form $y_p = u(t)y_h$, differentiating gives us

$$y_p{}' = u'y_h + uy_h{}'$$

and substituting into (4.3) results in

$$y_p{}' + p(t)y_p = u'y_h + uy_h{}' + p(t)uy_h = q(t).$$

Because $uy_h{}' + p(t)uy_h = u\left[y_h{}' + p(t)y_h\right] = u \cdot 0 = 0$, we obtain

y_h is a solution to the corresponding homogeneous equation so $y_h{}' + p(t)y_h = 0$.

$$\begin{aligned} u'y_h &= q(t) \\ u' &= \frac{1}{y_h}q(t) \\ u' &= e^{\int p(t)\,dt}q(t) \\ u &= \int e^{\int p(t)\,dt}q(t)\,dt \end{aligned}$$

so

$$y_p = u(t)\,y_h = Ce^{-\int p(t)\,dt}\int e^{\int p(t)\,dt}q(t)\,dt.$$

Because we can include an arbitrary constant of integration when evaluating $\int e^{\int p(t)\,dt}q(t)\,dt$, it follows that we can write a general solution of (4.3) as

$$y = e^{-\int p(t)\,dt}\int e^{\int p(t)\,dt}q(t)\,dt. \tag{4.5}$$

Thus, first-order linear equations can always be solved, although the resulting integrals may be difficult or impossible to evaluate exactly.

Mathematica is able to solve the general form of the first-order equation, the initial-value problem $y' + py = q$, $y(0) = y_0$,

```
In[575]:= DSolve[y′[t] + p[t]y[t] == q[t], y[t], t]
Out[575]= {{y[t] → e^(-∫_0^t p[DSolve‘t]ⅆDSolve‘t) C[1] + e^(-∫_0^t p[DSolve‘t]ⅆDSolve‘t)
              ∫_0^t e^(∫_0^DSolve‘t p[DSolve‘t]ⅆDSolve‘t) q[DSolve‘t]ⅆDSolve‘t}}
In[576]:= DSolve[{y′[t] + p[t]y[t] == q[t], y[0] == y0}, y[t], t]
Out[576]= {{y[t] → e^(-∫_0^t p[DSolve‘t]ⅆDSolve‘t) (y0+
              ∫_0^t e^(∫_0^DSolve‘t p[DSolve‘t]ⅆDSolve‘t) q[DSolve‘t]ⅆDSolve‘t)}}
```

as well as the corresponding homogeneous equation,

```
In[577]:= DSolve[y′[t] + p[t]y[t] == 0, y[t], t]
Out[577]= {{y[t] → e^(-∫_0^t p[DSolve‘t]ⅆDSolve‘t) C[1]}}

In[578]:= DSolve[{y′[t] + p[t]y[t] == 0, y[0] == y0}, y[t], t]
Out[578]= {{y[t] → e^(-∫_0^t p[DSolve‘t]ⅆDSolve‘t) y0}}
```

although the results contain unevaluated integrals.

Example 98 (Exponential Growth). Let $y = y(t)$ denote the size of a population at time t. If y grows at a rate proportional to the amount present, y satisfies

$$\frac{dy}{dt} = \alpha y, \tag{4.6}$$

where α is the **growth constant**. If $y(0) = y_0$, using (4.5) results in $y = y_0 e^{\alpha t}$. We use `DSolve` to confirm this result.

```
In[579]:= DSolve[{y'[t] == α y[t], y[0] == y0}, y[t], t]
Out[579]= {{y[t] → e^(t α) y0}}
```

$\frac{dy}{dt} = k(y - y_s)$ models *Newton's Law of Cooling*: the rate at which the temperature, $y(t)$, changes in a heating/cooling body is proportional to the difference between the temperature of the body and the constant temperature, y_s, of the surroundings.

Example 99. Solve each of the following equations: (a) $\frac{dy}{dt} = k(y - y_s)$, $y(0) = y_0$, k and y_s constant (b) $y' - 2ty = t$ (c) $ty' - y = 4t \cos 4t - \sin 4t$

Solution. By hand, we rewrite the equation and obtain

$$\frac{dy}{dt} - ky = -ky_s$$

A general solution of the corresponding homogeneous equation

$$\frac{dy}{dt} - ky = 0$$

is $y_h = e^{kt}$. Because k and $-ky_s$ are constants, we suppose that a particular solution of the nonhomogeneous equation, y_p, has the form $y_p = A$, where A is a constant.

This will turn out to be a lucky guess. If there is not a solution of this form, we will not be able to find it.

Assuming that $y_p = A$, we have $y_p' = 0$ and substitution into the nonhomogeneous equation gives us

$$\frac{dy_p}{dt} - ky_p = -KA = -ky_s \qquad \text{so} \qquad A = y_s.$$

Thus, a general solution is $y = y_h + y_p = Ce^{kt} + y_s$. Applying the initial condition $y(0) = y_0$ results in $y = y_s + (y_0 - y_s)e^{kt}$.

We obtain the same result with `DSolve`. We graph the solution satisfing $y(0) = 75$ assuming that $k = -1/2$ and $y_s = 300$ in Figure 4.5. Notice that $y(t) \to y_s$ as $t \to \infty$.

```
In[580]:= sola = DSolve[{y'[t] == k (y[t] - ys), y[0] == y0}, y[t], t]
Out[580]= {{y[t] → e^(k t) (y0 - ys) + ys}}

In[581]:= tp = sola[[1, 1, 2]]/.{k-> -1/2, ys-> 300, y0-> 75};
          Plot[tp, {t, 0, 10}]
```

(b) The equation is in standard form and we identify $p(t) = -2t$. Then, the integrating factor is $\mu(t) = e^{\int p(t),dt} = e^{-t^2}$. Multiplying the equation by the integrating factor, $\mu(t)$, results in

$$e^{-t^2}(y' - 2ty) = te^{-t^2} \qquad \text{or} \qquad \frac{d}{dt}\left(ye^{-t^2}\right) = te^{-t^2}$$

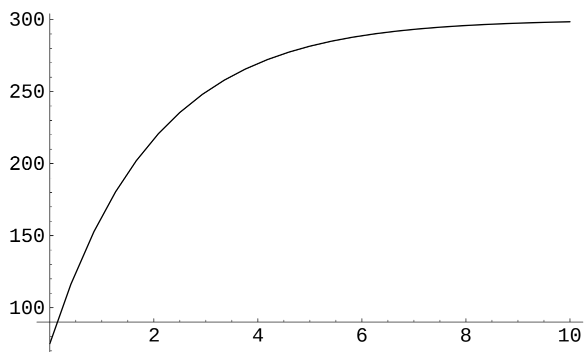


Figure 4.5: The temperature of the body approaches the temperature of its surroundings

Integrating gives us

$$ye^{-t^2} = -\frac{1}{2}e^{-t^2} + C \qquad \text{or} \qquad y = -\frac{1}{2} + Ce^{t^2}.$$

We confirm the result with `DSolve`.

```
In[582]:= DSolve[y'[t] - 2t y[t] == t, y[t], t]
Out[582]= {{y[t] → -1/2 + e^(t^2) C[1]}}
```

(c) In standard form, the equation is $y' - y/t = (4t \cos 4t - \sin 4t)/t$ so $p(t) = -1/t$. The integrating factor is $\mu(t) = e^{\int p(t),dt} = e^{-\ln t} = 1/t$ and multiplying the equation by the integrating factor and then integrating gives us

$$\begin{aligned}
\frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2}y &= \frac{1}{t^2}(4t\cos 4t - \sin 4t) \\
\frac{d}{dt}\left(\frac{1}{t}y\right) &= \frac{1}{t^2}(4t\cos 4t - \sin 4t) \\
\frac{1}{t}y &= \frac{\sin 4t}{t} + C \\
y &= \sin 4t + Ct,
\end{aligned}$$

where we use the `Integrate` function to evaluate $\int \frac{1}{t^2}(4t\cos 4t - \sin 4t)\,dt = \frac{\sin 4t}{t} + C$.

```
In[583]:= Integrate[(4 t Cos[4t] - Sin[4t])/t^2, t]
Out[583]= Sin[4 t]/t
```

We confirm this result with `DSolve`.

```
In[584]:= sol =
            DSolve[y'[t] - y[t]/t == (4 t Cos[4t] - Sin[4t])/t, y[t], t]
Out[584]= {{y[t] → t C[1] + Sin[4 t]}}
```

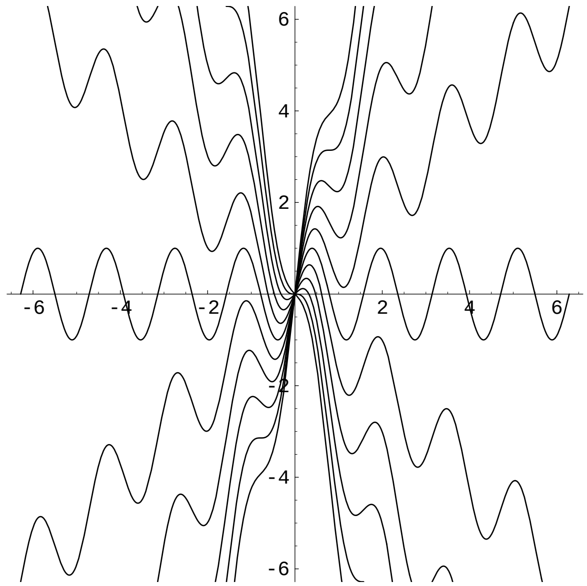


Figure 4.6: Every solution satisfies $y(0) = 0$

In the general solution, observe that *every* solution satisfies $y(0) = 0$. That is, the initial-value problem

$$\frac{dy}{dt} - \frac{1}{t}y = \frac{1}{t^2}(4t\cos 4t - \sin 4t), \qquad y(0) = 0$$

has infinitely many solutions. We see this in the plot of several solutions that is generated with `Plot` in Figure 4.6.

```
In[585]:= toplot = Table[sol[[1, 1, 2]]/.C[1]- > i, {i, -5, 5}];
          Plot[Evaluate[toplot], {t, -2π, 2π},
            PlotRange- > {-2π, 2π}, AspectRatio- > 1]
```

□

4.1.3 Nonlinear Equations

Mathematica can solve a variety of nonlinear first-order equations that are typically encountered in the introductory differential equations course.

Example 100. Solve each: (a) $(y\cos x + 2xe^y)\,dx + \left(\sin y + x^2e^y - 1\right)dy = 0$ (b) $\left(y^2 + 2xy\right)dx - x^2dy = 0$

Solution. (a) Notice that $(y\cos x + 2xe^y)\,dx + \left(\sin y + x^2e^y - 1\right)dy = 0$ can be written as $dy/dx = -(y\cos x + 2xe^y)/\left(\sin x + x^2e^y - 1\right)$.

The equation is an example of an *exact equation*. A theorem tells us that the equation

See your text for details.

$$M(x,y)dx + N(x,y)dy = 0$$

is **exact** if and only if $\partial M/\partial y = \partial N/\partial x$.

```
In[586]:= m = Cos[x] + 2 x Exp[y];
          n = Sin[y] + x^2 Exp[y] - 1;
          D[m, y]

          D[n, x]
Out[586]= 2 e^y x
Out[586]= 2 e^y x
```

We solve exact equations by integrating. Let $F(x,y) = C$ satisfy $(y\cos x + 2xe^y)\,dx + \left(\sin y + x^2e^y - 1\right)dy = 0$. Then,

$$F(x,y) = \int (\cos x + 2xe^y)\,dx = \sin x + x^2e^y + g(y),$$

where $g(y)$ is a function of y.

```
In[587]:= f1 = Integrate[m, x]
Out[587]= e^y x^2 + Sin[x]
```

We next find that $g'(y) = \sin y - 1$ so $g(y) = -\cos y - y$. Hence, a general solution of the equation is

$$\sin x + x^2e^y - \cos y = C.$$

```
In[588]:= f2 = D[f1, y]
Out[588]= e^y x^2

In[589]:= f3 = Solve[f2 + c == n, c]
Out[589]= {{c → -1 + Sin[y]}}

In[590]:= Integrate[f3[[1, 1, 2]], y]
Out[590]= -y - Cos[y]
```

We confirm this result with `DSolve`. Notice that Mathematica warns us that it cannot solve for y explicitly and returns the same implicit solution obtained by us.

```
In[591]:= mf = m/.y- > y[x];
          nf = n/.y- > y[x];
          sol = DSolve[mf + nf y'[x] == 0, y[x], x]
Solve :: "tdep" : "Theequationsappeartoinvolve
     transcendentalfunctionsofthevariablesin
     anessentiallynon - algebraicway."
Out[591]= Solve[e^y[x] x^2 - Cos[y[x]] + Sin[x] - y[x] == C[1], {y[x]}]
```

Graphs of several solutions are graphed with `ContourPlot` in Figure 4.7.

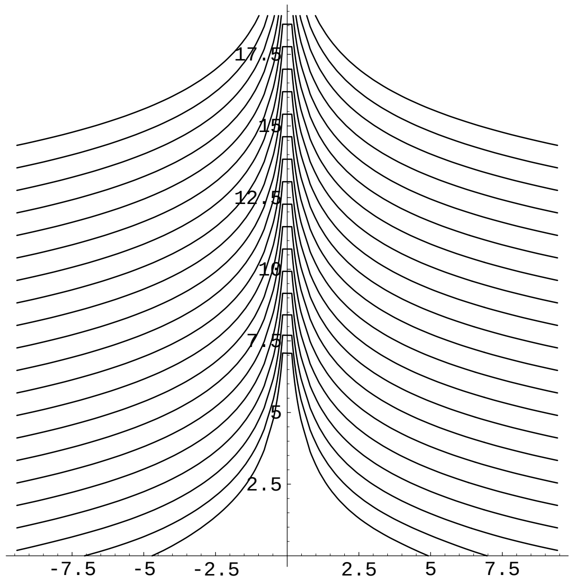


Figure 4.7: Graphs of several solutions of $(y\cos x + 2xe^y)\,dx + \left(\sin y + x^2 e^y - 1\right)dy = 0$

```
In[592]:= sol[[1, 1]]
Out[592]= e^y[x] x^2 - Cos[y[x]] + Sin[x] - y[x]

In[593]:= sol2 = sol[[1, 1]]/.y[x]-> y
Out[593]= e^y x^2 - y - Cos[y] + Sin[x]

In[594]:= cvals = Table[
              sol2/.{x-> -3π/2, y-> i}, {i, 0, 6π, 6π/24}]//
            N
Out[594]= {22.2066,
            48.2128, 106.254,
            233.647, 512.735,
            1124.85, 2468.28,
            5416.56, 11885.2,
            26074.5, 57196.7,
            125457., 275169.,
            603531., 1.32372 10^6,
            2.9033 10^6, 6.36776 10^6,
            1.39663 10^7, 3.0632 10^7,
            6.71846 10^7, 1.47355 10^8,
            3.2319 10^8, 7.08847 10^8,
            1.5547 10^9, 3.40989 10^9}

In[595]:= ContourPlot[sol2, {x, -3π, 3π}, {y, 0, 6π},
            ContourShading-> False, Frame-> False, Axes-> Automatic,
            AxesOrigin-> {0, 0}, Contours-> cvals, PlotPoints-> 60]
```

(b) We can write $\left(y^2+2xy\right)dx - x^2 dy = 0$ as $dy/dx = \left(y^2+2xy\right)/x^2$. A first-order equation is **homogeneous** if it can be written in the form

$$dy/dx = F(y/x).$$

Homogeneous equations are reduced to separable equations with either the substitution $y = ux$ or $x = vy$.

In this case, we have that $dy/dx = (y/x)^2 + 2(y/x)$ so the equation is homogeneous.

Let $y = ux$. Then, $dy = u\,dx + x\,du$. Substituting into $\left(y^2+2xy\right)dx - x^2 dy = 0$ and separating gives us

$$\begin{aligned}\left(y^2+2xy\right)dx - x^2 dy &= 0\\ \left(u^2x^2+2ux^2\right)dx - x^2(u\,dx + x\,du) &= 0\\ \left(u^2+2u\right)dx - (u\,dx + x\,du) &= 0\\ \left(u^2+u\right)dx &= -x\,du\\ \frac{1}{u(u+1)}du &= -\frac{1}{x}dx.\end{aligned}$$

Integrating the left and right-hand sides of this equation with `Integrate`,

```
In[596]:= Integrate[1/(u(u+1)), u]
Out[596]= Log[u] - Log[1 + u]
```

```
In[597]:= Integrate[-1/x, x]
Out[597]= -Log[x]
```

exponentiating, resubstituting $u = y/x$, and solving for y gives us

$$\begin{aligned}\ln|u| - \ln|u+1| &= -\ln|x| + C\\ \frac{u}{u+1} &= C/x\\ \frac{y/x}{y/x+1} &= C/x\\ y &= \frac{Cx}{x-C}.\end{aligned}$$

```
In[598]:= Solve[(y/x)/(y/x + 1) == c/x, y]
Out[598]= {{y -> -(c x)/(c - x)}}
```

We confirm this result with `DSolve` and then graph several solutions with `Plot` in Figure 4.8.

```
In[599]:= sol = DSolve[y[x]^2 + 2x y[x] - x^2y'[x] == 0, y[x], x]
Out[599]= {{y[x] -> -(x^2 C[1])/(-1 + x C[1])}}
```

```
In[600]:= toplot = Table[sol[[1, 1, 2]]/.C[1]- > i, {i, -5, 5}];
          Plot[Evaluate[toplot], {x, -5, 5}, PlotRange- > {-5, 5},
            AspectRatio- > Automatic]
```

□

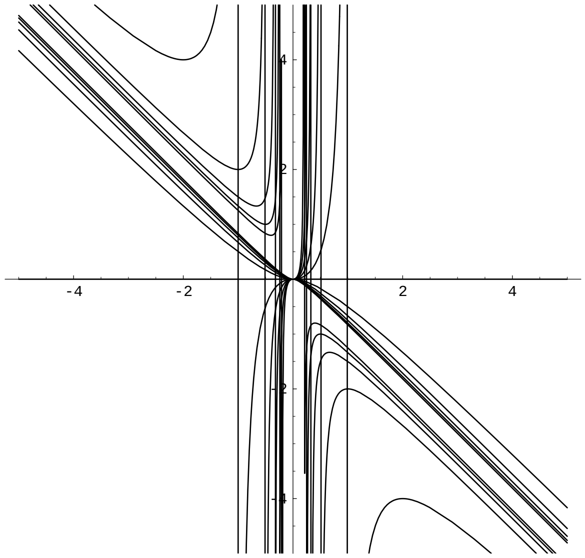


Figure 4.8: Graphs of several solutions of $\left(y^2+2xy\right)dx - x^2 dy = 0$

4.1.4 Numerical Methods

If numerical results are desired, use `NDSolve` (see Figure 4.9):

```
NDSolve[{y'[t]==f[t,y[t]],y[t0]==y0},y[t],{t,a,b}]
```

attempts to generate a numerical solution of

$$\begin{cases} \frac{dy}{dt} = f(t,y) \\ y\left(t_0\right) = y_0 \end{cases}$$

valid for $a \le t \le b$.

Example 101. Consider

$$\frac{dy}{dt} = \left(t^2 - y^2\right)\sin y,\ y(0) = -1.$$

(a) Determine $y(1)$. (b) Graph $y(t)$, $-1 \le t \le 10$.

Solution. We first remark that `DSolve` can neither exactly solve the differential equation $y' = \left(t^2 - y^2\right)\sin y$ nor find the solution that satisfies $y(0) = -1$.

```
In[601]:= sol = DSolve[y'[t] == (t^2 - y[t]^2) Sin[t], y[t], t]
Out[601]= BoxData(DSolve[y'[t] == Sin[t] (t^2 - y[t]^2), y[t], t])
```

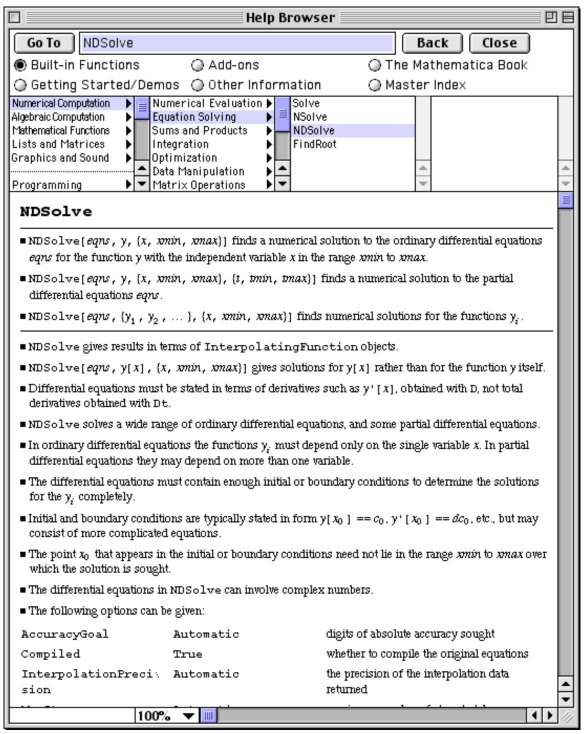


Figure 4.9: Using `NDSolve`

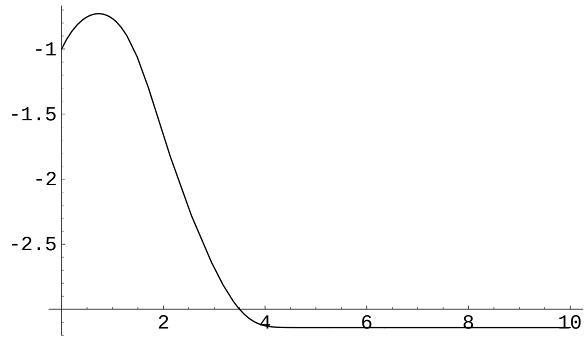


Figure 4.10: Graph of the solution to $y' = \left(t^2 - y^2\right)\sin y$, $y(0) = -1$

```
In[602]:= sol =
            DSolve[{y′[t] == (t^2 - y[t]^2) Sin[t], y[0] == y0}, y[t], t]
Out[602]= BoxData(DSolve[{y′[t] == Sin[t] (t^2 - y[t]^2), y[0] == y0}, y[t], t])
```

However, we obtain a numerical solution valid for $0 \leq t \leq 1000$ using the `NDSolve` function.

```
In[603]:= sol = NDSolve[{y′[t] == (t^2 - y[t]^2) Sin[y[t]], y[0] == -1},
              y[t], {t, 0, 1000}]
Out[603]= BoxData({{y[t] → InterpolatingFunction[{{0., 1000.}}, "<>"][t]}})
```

Entering `sol /.t->1` evaluates the numerical solution if $t = 1$.

```
In[604]:= sol /.t- > 1
Out[604]= {{y[1] → -0.766014}}
```

The result means that $y(1) \approx -.766$. We use `Plot` command to graph the solution for $0 \leq t \leq 10$ in Figure 4.10.

```
In[605]:= Plot[Evaluate[y[t]/.sol], {t, 0, 10}]
```

□

Example 102 (Logistic Equation with Predation). Incorporating predation into the **logistic equation**, $y' = \alpha y\left(1 - \frac{1}{K}y\right)$, results in

$$\frac{dy}{dt} = \alpha y\left(1 - \frac{1}{K}y\right) - P(y),$$

where $P(y)$ is a function of y describing the rate of predation. A typical choice for P is $P(y) = ay^2/(b^2 + y^2)$ because $P(0) = 0$ and P is bounded above: $\lim_{t\to\infty} P(y) < \infty$.

Remark. Of course, if $\lim_{t\to\infty} y(t) = Y$, then $\lim_{t\to\infty} P(y) = aY^2/(b^2 + Y^2)$. Generally, however, $\lim_{t\to\infty} P(y) \neq a$ because $\lim_{t\to\infty} y(t) \leq K \neq \infty$, for some $K \geq 0$, in the predation situation.

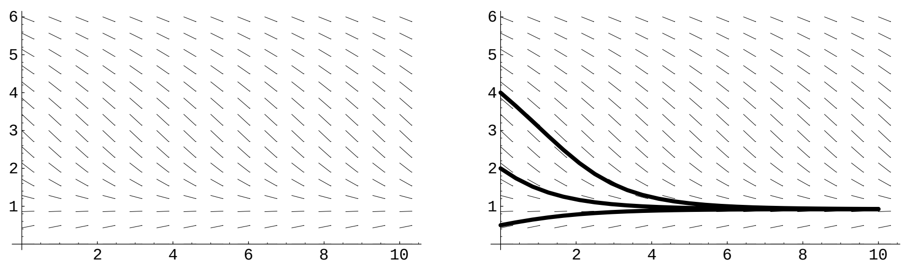


Figure 4.11: (a) Direction field and (b) direction field with three solutions

If $\alpha = 1$, $a = 5$ and $b = 2$, graph the direction field associated with the equation as well as various solutions if (a) $K = 19$ and (b) $K = 20$.

Solution. (a) We define `eqn[k]` to be

$$\frac{dy}{dt} = y\left(1 - \frac{1}{K}y\right) - \frac{5y^2}{4+y^2}.$$

```
In[606]:= << Graphics`PlotField`

In[607]:= eqn[k_] = y'[t] == y[t](1 - 1/k y[t]) - 5y[t]^2/(4 + y[t]^2);
```

We use `PlotVectorField` to graph the direction field in Figure 4.11 (a) and then the direction field along with the solutions that satisfy $y(0) = .5$, $y(0) = .2$, and $y(0) = 4$ in Figure 4.11 (b).

```
In[608]:= pvf19 = PlotVectorField[{1, y(1 - 1/19 y) - 5y^2/(4 + y^2)},
              {t, 0, 10}, {y, 0, 6}, Axes- > Automatic, HeadLength- > 0,
              DisplayFunction- > Identity];

In[609]:= n1 = NDSolve[{eqn[19], y[0] == 0.5}, y[t], {t, 0, 10}];
          n2 = NDSolve[{eqn[19], y[0] == 2}, y[t], {t, 0, 10}];
          n3 = NDSolve[{eqn[19], y[0] == 4}, y[t], {t, 0, 10}];

In[610]:= solplot = Plot[Evaluate[y[t]/.{n1, n2, n3}],
              {t, 0, 10}, PlotStyle- > Thickness[0.01],
              DisplayFunction- > Identity];
```

The same results can be obtained using `Map`.

```
In[611]:= numsols = Map[NDSolve[
                  {eqn[19], y[0] == #}, y[t], {t, 0, 10}]&, {0.5, 2, 4}];
          solplot = Plot[Evaluate[y[t]/.numsols],
              {t, 0, 10}, PlotStyle- > Thickness[0.01],
              DisplayFunction- > Identity];

In[612]:= Show[GraphicsArray[{pvf19, Show[pvf19, solplot]}]]
```

In the plot, notice that all nontrivial solutions appear to approach an equilibrium solution. We determine the equilibrium solution by solving $y' = 0$

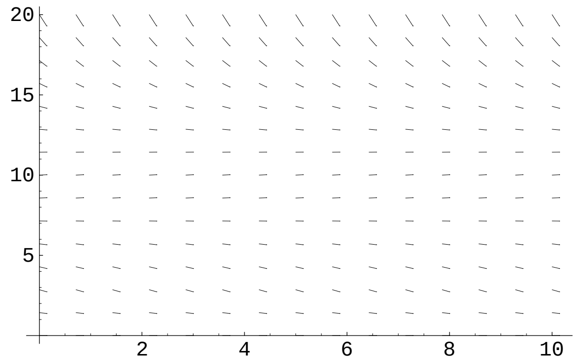


Figure 4.12: Direction field

```
In[613]:= eqn[19][[2]]
```

Out[613]= $\left(1-\frac{y[t]}{19}\right)\ y[t]-\frac{5\ y[t]^2}{4+y[t]^2}$

```
In[614]:= Solve[eqn[19.][[2]] == 0, y[t]]
Out[614]= {{y[t] → 0.}, {y[t] → 0.923351},
           {y[t] → 9.03832 − 0.785875 i},
           {y[t] → 9.03832 + 0.785875 i}}
```

to see that it is $y \approx 0.923$.

(b) We carry out similar steps for (b). First, we graph the direction field with `PlotVectorField` in Figure 4.12.

```
In[615]:= pvf20 = PlotVectorField[{1, y(1 - 1/20 y) - 5y^2/(4 + y^2)},
             {t, 0, 10}, {y, 0, 20}, Axes- > Automatic,
             HeadLength- > 0, AspectRatio- > 1/GoldenRatio];
```

We then use `Map` together with NDSolve to numerically find the solution satisfying $y(0) = .5i$, for $i = 1, 2, \ldots, 40$ and name the resulting list `numsols`. The functions contained in `numsols` are graphed with `Plot` in `solplot`.

```
In[616]:= numsols =
            Map[NDSolve[{eqn[20], y[0] == #}, y[t], {t, 0, 10}]&,
             Table[0.5i, {i, 1, 40}]];
          solplot = Plot[Evaluate[y[t]/.numsols],
             {t, 0, 10}, PlotStyle- > Thickness[0.005],
             DisplayFunction- > Identity];
```

Last, we display the direction field along with the solution graphs in `solplot` using `Show` in Figure 4.13.

```
In[617]:= Show[pvf20, solplot]
```

Notice that there are three nontrivial equilibrium solutions that are found by solving $y' = 0$.

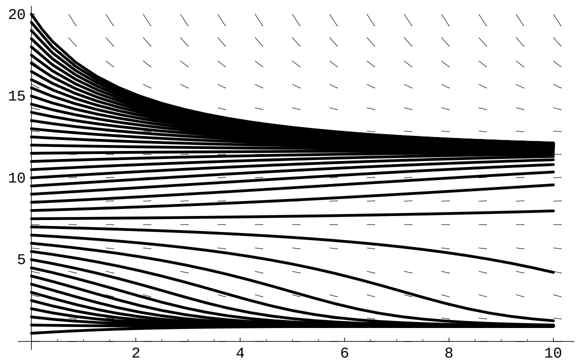


Figure 4.13: Direction field with several solutions

```
In[618]:= Solve[eqn[20.][[2]] == 0, y[t]]
Out[618]= {{y[t] → 0.}, {y[t] → 0.926741},
           {y[t] → 7.38645},
           {y[t] → 11.6868}}
```

In this case, $y \approx .926$ and $y \approx 11.687$ are stable while $y \approx 7.386$ is unstable. □

4.2 Second-Order Linear Equations

We now present a concise discussion of second-order linear equations, which are extensively discussed in the introductory differential equations course.

4.2.1 Basic Theory

The **general form** of the **second-order linear equation** is

$$a_2(t)\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t), \tag{4.7}$$

where $a_2(t)$ is not identically the zero function.

The **standard form** of the second-order linear equation (4.7) is

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t). \tag{4.8}$$

The **corresponding homogeneous equation** of (4.8) is

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0. \tag{4.9}$$

A **general solution** of (4.9) is $y = c_1y_1 + c_2y_2$ where

1. y_1 and y_2 are solutions of (4.9), and
2. y_1 and y_2 are *linearly independent*.

If y_1 and y_2 are solutions of (4.9), then y_1 and y_2 are **linearly independent** if and only if the **Wronskian**,

$$W\left(\{y_1, y_2\}\right) = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = y_1 y_2{}' - y_1{}' y_2, \tag{4.10}$$

is not the zero function. If y_1 and y_2 are linearly independent solutions of (4.9), we call the set $S = \{y_1, y_2\}$ a **fundamental set of solutions** for (4.9).

A particular solution, y_p, is a solution that does not contain any arbitrary constants.

Let y be a general solution of (4.8) and y_p be a particular solution of (4.8). It follows that $y - y_p$ is a solution of (4.9) so $y - y_p = y_h$ where y_h is a general solution of (4.9). Hence, $y = y_h + y_p$. That is, to solve the nonhomogeneous equation, we need a general solution, y_h, of the corresponding homogeneous equation and a particular solution, y_p, of the nonhomogeneous equation.

4.2.2 Constant Coefficients

Suppose that the coefficient functions of (4.7) are constants: $a_2(t) = a$, $a_1(t) = b$, and $a_0(t) = c$ and that $f(t)$ is identically the zero function. In this case, (4.7) becomes

$$ay'' + by' + cy = 0. \tag{4.11}$$

Now suppose that $y = e^{kt}$, k constant, is a solution of (4.11). Then, $y' = ke^{kt}$ and $y'' = k^2 e^{kt}$. Substitution into (4.11) then gives us

$$\begin{aligned} ay'' + by' + cy &= ak^2 e^{kt} + bke^{kt} + ce^{kt} \\ &= e^{kt}\left(ak^2 + bk + c\right) = 0. \end{aligned}$$

Because $e^{kt} \neq 0$, the solutions of (4.11) are determined by the solutions of

$$ak^2 + bk + c = 0, \tag{4.12}$$

called the **characteristic equation** of (4.11).

Theorem 22. *Let k_1 and k_2 be the solutions of* (4.12).

1. *If $k_1 \neq k_2$ are real and distinct, two linearly independent solutions of* (4.11) *are $y_1 = e^{k_1 t}$ and $y_2 = e^{k_2 t}$; a general solution of* (4.11) *is*

$$y = c_1 e^{k_1 t} + c_2 e^{k_2 t}.$$

2. *If $k_1 = k_2$, two linearly independent solutions of* (4.11) *are $y_1 = e^{k_1 t}$ and $y_2 = te^{k_1 t}$; a general solution of* (4.11) *is*

$$y = c_1 e^{k_1 t} + c_2 t e^{k_1 t}.$$

3. *If* $k_{1,2} = \alpha \pm \beta i$, $\beta \neq 0$, *two linearly independent solutions of* (4.11) *are* $y_1 = e^{\alpha t}\cos\beta t$ *and* $y_2 = e^{\alpha t}\sin\beta t$; *a general solution of* (4.11) *is*

$$y = e^{\alpha t}\left(c_1\cos\beta t + c_2\sin\beta t\right).$$

Example 103. Solve each of the following equations. (a) $6y'' + y' - 2y = 0$ (b) $y'' + 2y' + y = 0$ (c) $16y'' + 8y' + 145y = 0$

Solution. (a) The characteristic equation is $6k^2 + k - 2 = (3k+2)(2k-1) = 0$ with solutions $k = -2/3$ and $k = 1/2$. We check with either `Factor` or `Solve`.

```
In[619]:= Factor[6k^2 + k - 2]

          Solve[6k^2 + k - 2 == 0]
Out[619]= (-1 + 2 k) (2 + 3 k)
Out[619]= {{k -> -2/3}, {k -> 1/2}}
```

Then, a fundamental set of solutions is $\left\{e^{-2t/3}, e^{t/2}\right\}$ and a general solution is

$$y = c_1 e^{-2t/3} + c_2 e^{t/2}.$$

Of course, we obtain the same result with `DSolve`.

```
In[620]:= DSolve[6y''[t] + y'[t] - 2y[t] == 0, y[t], t]
Out[620]= {{y[t] -> e^(-2 t/3) C[1] + e^(t/2) C[2]}}
```

(b) The characteristic equation is $k^2 + 2k + 1 = (k+1)^2 = 0$ with solution $k = -1$, which has multiplicity two, so a fundamental set of solutions is $\left\{e^{-t}, te^{-t}\right\}$ and a general solution is

$$y = c_1 e^{-t} + c_2 t e^{-t}.$$

We check the calculation in the exact same way as in (a).

```
In[621]:= Factor[k^2 + 2k + 1]

          Solve[k^2 + 2k + 1 == 0]

          DSolve[y''[t] + 2y'[t] + y[t] == 0, y[t], t]
Out[621]= (1 + k)^2
Out[621]= {{k -> -1}, {k -> -1}}
Out[621]= {{y[t] -> e^-t C[1] + e^-t t C[2]}}
```

(c) The characteristic equation is $16k^2 + 8k + 145 = 0$ with solutions $k_{1,2} = -\frac{1}{4} \pm 3i$ so a fundamental set of solutions is $\left\{e^{-t/4}\cos 3t, e^{-t/4}\sin 3t\right\}$ and a general solution is

$$y = e^{-t/4}\left(c_1\cos 3t + c_2\sin 3t\right).$$

The calculation is verified in the same way as in (a) and (b).

```
In[622]:= Factor[16k^2 + 8k + 145, GaussianIntegers- > True]

          Solve[16k^2 + 8k + 145 == 0]

          DSolve[16y''[t] + 8y'[t] + 145y[t] == 0, y[t], t]
Out[622]= ((1 - 12 i) + 4 k) ((1 + 12 i) + 4 k)
Out[622]= {{k -> -1/4 - 3 i}, {k -> -1/4 + 3 i}}
Out[622]= {{y[t] -> e^(-t/4) C[2] Cos[3 t] - e^(-t/4) C[1] Sin[3 t]}}
```

□

Example 104. Solve

$$64\frac{d^2y}{dt^2} + 16\frac{dy}{dt} + 1025y = 0,\ y(0) = 1,\ \frac{dy}{dt}(0) = 2.$$

Solution. A general solution of $64y''+16y'+1025y = 0$ is $y = e^{-t/8}\left(c_1 \sin 4t + c_2 \cos 4t\right)$.

```
In[623]:= gensol = DSolve[64y''[t] + 16y'[t] + 1025y[t] == 0, y[t], t]
Out[623]= {{y[t] -> e^(-t/8) C[2] Cos[4 t] - e^(-t/8) C[1] Sin[4 t]}}
```

Applying $y(0) = 1$ shows us that $c_2 = 1$.

```
In[624]:= e1 = y[t]/.gensol[[1]]/.t- > 0
Out[624]= C[2]
```

Computing y'

```
In[625]:= D[y[t]/.gensol[[1]], t]
Out[625]= -4 e^(-t/8) C[1] Cos[4 t] - 1/8 e^(-t/8) C[2] Cos[4 t]+
            1/8 e^(-t/8) C[1] Sin[4 t] - 4 e^(-t/8) C[2] Sin[4 t]
```

and then $y'(0)$, shows us that $4c_1 - \frac{1}{8}c_2$=2.

```
In[626]:= e2 = D[y[t]/.gensol[[1]], t]/.t- > 0
Out[626]= -4 C[1] - C[2]/8
```

Solving for c_1 and c_2 with `Solve` shows us that $c_1 = \frac{17}{32}$ and $c_1 = 1$.

```
In[627]:= cvals = Solve[{e1 == 1, e2 == 3}]
Out[627]= {{C[1] -> -25/32, C[2] -> 1}}
```

Thus, $y = e^{-t/8}\left(\frac{17}{32} \sin 4t + \cos 4t\right)$, which we graph with `Plot` in Figure 4.14.

```
In[628]:= sol = y[t]/.gensol[[1]]/.cvals[[1]]
Out[628]= e^(-t/8) Cos[4 t] + 25/32 e^(-t/8) Sin[4 t]
In[629]:= Plot[sol, {t, 0, 8π}]
```

We verify the calculation with `DSolve`.

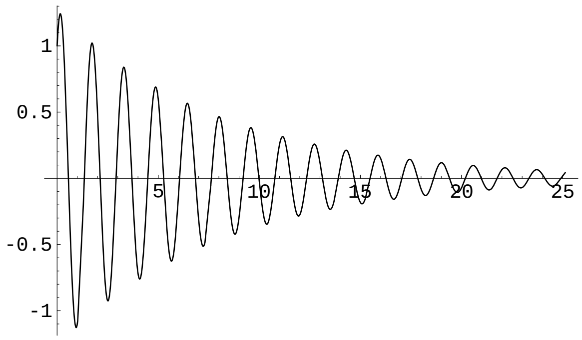


Figure 4.14: The solution to the initial-value problem tends to 0 as $t \to \infty$

```
In[630]:= DSolve[
           {64y''[t] + 16y'[t] + 1025y[t] == 0, y[0] == 1, y'[0] == 2},
           y[t], t]
Out[630]= {{y[t] → e^(-t/8) (Cos[4 t] + 17/32 Sin[4 t])}}
```

□

4.2.3 Undetermined Coefficients

If (4.7) has constant coefficients and $f(t)$ is a product of terms t^n, $e^{\alpha t}$, α constant, $\cos\beta t$, and/or $\sin\beta t$, β constant, *undetermined coefficients* can often be used to find a particular solution of (4.7). The key to implementing the method is to *judiciously* choose the correct form of y_p.

Assume that a general solution, y_h, of the corresponding homogeneous equation has been found and that each term of $f(t)$ has the form

$$t^n e^{\alpha t}\cos\beta t \qquad \text{or} \qquad t^n e^{\alpha t}\cos\beta t.$$

For *each* term of $f(t)$, write down the *associated set*

$$F = \left\{t^n e^{\alpha t}\cos\beta t, t^n e^{\alpha t}\sin\beta t, t^{n-1}e^{\alpha t}\cos\beta t, t^{n-1}e^{\alpha t}\sin\beta t, \ldots, e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t, \right\}.$$

If any element of F is a solution to the corresponding homogeneous equation, multiply each element of F by t^m, where m is the smallest positive integer so that none of the elements of $t^m F$ are solutions to the corresponding homogeneous equation. A particular solution will be a linear combination of the functions in all the F's.

Example 105. Solve

$$4\frac{d^2y}{dt^2} - y = t - 2 - 5\cos t - e^{-t/2}.$$

Solution. The corresponding homogeneous equation is $4y'' - y = 0$ with general solution $y_h = c_1 e^{-t/2} + c_2 e^{t/2}$.

```
In[631]:= DSolve[4y''[t] - y[t] == 0, y[t], t]
Out[631]= {{y[t] → e^(-t/2) C[1] + e^(t/2) C[2]}}
```

A fundamental set of solutions for the corresponding homogeneous equation is $S = \left\{e^{-t/2}, e^{t/2}\right\}$. The associated set of functions for $t - 2$ is $F_1 = \{1, t\}$, the associated set of functions for $-5\cos t$ is $F_2 = \{\cos t, \sin t\}$, and the associated set of functions for $-e^{-t/2}$ is $F_3 = \left\{e^{-t/2}\right\}$. Note that $e^{-t/2}$ is an element of S so we multiply F_3 by t resulting in $tF_3 = \left\{te^{-t/2}\right\}$.

No element of F_1 is contained in S and no element of F_2 is contained in S.

Then, we search for a particular solution of the form

$$y_p = A + Bt + C\cos t + D\sin t + Ete^{-t/2},$$

where A, B, C, D, and E are constants to be determined.

```
In[632]:= yp[t_] = a + b t + c Cos[t] + d Sin[t] + e t Exp[-t/2]
Out[632]= a + b t + e e^(-t/2) t + c Cos[t] + d Sin[t]
```

Computing y_p' and y_p''

```
In[633]:= dyp = yp'[t]

          d2yp = yp''[t]
Out[633]= b + e e^(-t/2) - 1/2 e e^(-t/2) t + d Cos[t] - c Sin[t]
Out[633]= -e e^(-t/2) + 1/4 e e^(-t/2) t - c Cos[t] - d Sin[t]
```

and substituting into the nonhomogeneous equation results in

$$-A - Bt - 5C\cos t - 5D\sin t - 4Ee^{-t/2} = t - 2 - 5\cos t - e^{-t/2}.$$

```
In[634]:= eqn = 4 yp''[t] - yp[t] == t - 2 - 5Cos[t] - Exp[-t/2]
Out[634]= -a - b t - e e^(-t/2) t - c Cos[t] - d Sin[t] +
            4 (-e e^(-t/2) + 1/4 e e^(-t/2) t - c Cos[t] - d Sin[t]) ==
           -2 - e^(-t/2) + t - 5 Cos[t]
```

Equating coefficients results in

$$-A = -2 \qquad -B = 1 \qquad -5C = -5 \qquad -5D = 0 \qquad -4E = -1$$

so $A = 2$, $B = -1$, $C = 1$, $D = 0$, and $E = 1/4$.

```
In[635]:= cvals =
            Solve[{-a == -2, -b == 1, -5c == -5, -5d == 0, -4e == -1}]
Out[635]= {{a → 2, b → -1, c → 1, d → 0, e → 1/4}}
```

y_p is then given by $y_p = 2 - t + \cos t + \frac{1}{4}te^{-t/2}$

```
In[636]:= yp[t]/.cvals[[1]]
```

Out[636]= $2-t+\frac{1}{4}e^{-t/2}t+\text{Cos}[t]$

and a general solution is given by

$$y=y_h+y_p=c_1e^{-t/2}+c_e^{t/2}+2-t+\cos t+\frac{1}{4}te^{-t/2}.$$

Note that $-A-Bt-5C\cos t-5D\sin t-4Ee^{-t/2}=t-2-5\cos t-e^{-t/2}$ is true for *all* values of t. Evaluating for five different values of t gives us five equations that we then solve for A, B, C, D, and E, resulting in the same solutions as already obtained.

```
In[637]:= e1 = eqn/.t- > 0
```

Out[637]= $-a-c+4(-c-e)==-8$

```
In[638]:= e2 = eqn/.t- > π/2

          e3 = eqn/.t- > π

          e4 = eqn/.t- > 1

          e5 = eqn/.t- > 2
```

Out[638]= $-a-d-\frac{b\pi}{2}-\frac{1}{2}e\,e^{-\pi/4}\pi+4\left(-d-e\,e^{-\pi/4}+\frac{1}{8}e\,e^{-\pi/4}\pi\right)==-2-e^{-\pi/4}+\frac{\pi}{2}$

Out[638]= $-a+c-b\pi-e\,e^{-\pi/2}\pi+4\left(c-e\,e^{-\pi/2}+\frac{1}{4}e\,e^{-\pi/2}\pi\right)==3-e^{-\pi/2}+\pi$

Out[638]= $-a-b-\frac{e}{\sqrt{e}}-c\,\text{Cos}[1]-d\,\text{Sin}[1]+4\left(-\frac{3e}{4\sqrt{e}}-c\,\text{Cos}[1]-d\,\text{Sin}[1]\right)==-1-\frac{1}{\sqrt{e}}-5\,\text{Cos}[1]$

Out[638]= $-a-2b-\frac{2e}{e}-c\,\text{Cos}[2]-d\,\text{Sin}[2]+4\left(-\frac{e}{2e}-c\,\text{Cos}[2]-d\,\text{Sin}[2]\right)==-\frac{1}{e}-5\,\text{Cos}[2]$

```
In[639]:= Solve[{e1, e2, e3, e4, e5}, {a, b, c, d, e}]//Simplify
```

Out[639]= $\{\{d\to 0, b\to -1, a\to 2, c\to 1, e\to \frac{1}{4}\}\}$

Last, we check our calculation with `DSolve` and `simplify`.

```
In[640]:= sol2 =
            DSolve[4y''[t] - y[t] == t - 2 - 5Cos[t] - Exp[-t/2], y[t], t]
```

Out[640]= $\{\{y[t]\to e^{-t/2}C[1]+e^{t/2}C[2]+\frac{1}{4}\left(e^{-t/2}-2t+2\,\text{Cos}[t]-4\,\text{Sin}[t]\right)+e^{-t/2}\left(2e^{t/2}+\frac{t}{4}-\frac{1}{2}e^{t/2}t+\frac{1}{2}e^{t/2}\text{Cos}[t]+e^{t/2}\text{Sin}[t]\right)\}\}$

```
In[641]:= Simplify[sol2]
```

```
Out[641]= {{y[t] →
            1/4 e^(-t/2) (1 + 8 e^(t/2) + t - 4 e^(t/2) t + 4 C[1] + 4 e^t C[2]) + Cos[t]}}
```

□

Example 106. Solve $y'' + 4y = \cos 2t$, $y(0) = 0$, $y'(0) = 0$.

Solution. A general solution of the corresponding homogeneous equation is $y_h = c_1 \cos 2t + c_2 \sin 2t$. For this equation, $F = \{\cos 2t, \sin 2t\}$. Because elements of F are solutions to the corresponding homogeneous equation, we multiply each element of F by t resulting in $tF = \{t \cos 2t, t \sin 2t\}$. Therefore, we assume that a particular solution has the form

$$y_p = At \cos 2t + Bt \sin 2t,$$

where A and B are constants to be determined. Proceeding in the same manner as before, we compute y_p' and y_p''

```
In[642]:= yp[t_] = a t Cos[2t] + b t Sin[2 t];
          yp'[t]

          yp''[t]
Out[642]= a Cos[2 t] + 2 b t Cos[2 t] + b Sin[2 t] - 2 a t Sin[2 t]
Out[642]= 4 b Cos[2 t] - 4 a t Cos[2 t] - 4 a Sin[2 t] - 4 b t Sin[2 t]
```

and then substitute into the nonhomogeneous equation

```
In[643]:= eqn = yp''[t] + 4yp[t] == Cos[2t]
Out[643]= 4 b Cos[2 t] - 4 a t Cos[2 t] - 4 a Sin[2 t]-
            4 b t Sin[2 t] + 4 (a t Cos[2 t] + b t Sin[2 t]) ==
           Cos[2 t]
```

Equating coefficients readily yields $A = 0$ and $B = 1/4$. Alternatively, remember that $-4A \sin 2t + 4B \cos 2t = \cos 2t$ is true for *all* values of t. Evaluating for two values of t and then solving for A and B gives the same result.

```
In[644]:= e1 = eqn/.t- > 0

          e2 = eqn/.t- > π/4

          cvals = Solve[{e1, e2}]
Out[644]= 4 b == 1
Out[644]= -4 a == 0
Out[644]= {{a → 0, b → 1/4}}
```

It follows that $y_p = \frac{1}{4} t \sin 2t$ and $y = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4} t \sin 2t$.

```
In[645]:= yp[t]/.cvals[[1]]
Out[645]= 1/4 t Sin[2 t]
In[646]:= y[t_] = c1 Cos[2t] + c2 Sin[2t] + 1/4 t Sin[2t]
Out[646]= c1 Cos[2 t] + c2 Sin[2 t] + 1/4 t Sin[2 t]
```

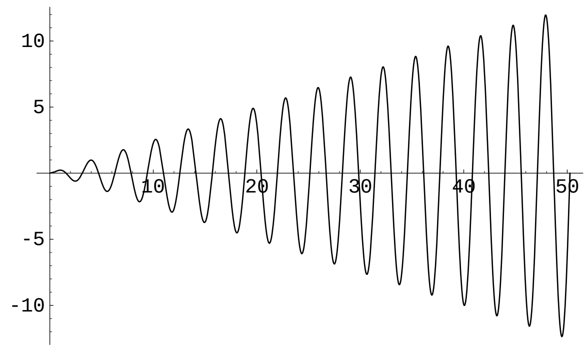


Figure 4.15: The forcing function causes the solution to become unbounded as $t \to \infty$

Applying the initial conditions

```
In[647]:= y'[t]
Out[647]= 2 c2 Cos[2 t] + 1/2 t Cos[2 t] + 1/4 Sin[2 t] - 2 c1 Sin[2 t]
In[648]:= cvals = Solve[{y[0] == 0, y'[0] == 0}]
Out[648]= {{c1 → 0, c2 → 0}}
```

results in $y = \frac{1}{4} t \sin 2t$, which we graph with `Plot` in Figure 4.15.

```
In[649]:= y[t]/.cvals[[1]]
Out[649]= 1/4 t Sin[2 t]
In[650]:= Plot[Evaluate[y[t]/.cvals[[1]]], {t, 0, 16π}]
```

We verify the calculation with `DSolve`.

```
In[651]:= Clear[y]

          DSolve[
            y''[t] + 4y[t] == Cos[2t], y[0] == 0, y'[0] == 0}, y[t], t]
Out[651]= {{y[t] → 1/4 t Sin[2 t]}}
```

□

4.2.4 Variation of Parameters

Let $S = \{y_1, y_2\}$ be a fundamental set of solutions for (4.9). To solve the nonhomogeneous equation (4.8), we need to find a particular solution, y_p of (4.8). We search for a particular solution of the form

A particular solution, y_p, is a solution that does not contain any arbitrary constants.

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t), \tag{4.13}$$

where u_1 and u_2 are functions of t. Differentiating (4.13) gives us

Observe that it is pointless to search for solutions of the form $y_p = c_1y_1 + c_2y_2$ where c_1 and c_2 are constants because for every choice of c_1 and c_2, $c_1y_1 + c_2y_2$ is a solution to the corresponding homogeneous equation.

$$y_p{}' = u_1{}'y_1 + u_1y_1{}' + u_2{}'y_2 + u_2y_2{}'.$$

Assuming that

$$y_1u_1{}' + y_2u_2{}' = 0 \tag{4.14}$$

results in $y_p{}' = u_1y_1{}' + u_2y_2{}'$. Computing the second derivative then yields

$$y_p{}'' = u_1{}'y_1{}' + u_1y_1{}'' + u_2{}'y_2{}' + u_2y_2{}''.$$

Substituting y_p, $y_p{}'$, and $y_p{}''$ into (4.8) and using the facts that

$$u_1\left(y_1{}'' + p\,y_1{}' + q\,y_1\right) = 0 \quad \text{and} \quad u_2\left(y_2{}'' + p\,y_2{}' + q\,y_2\right) = 0$$

(because y_1 and y_2 are solutions to the corresponding homogeneous equation) results in

$$\begin{aligned}\frac{d^2y_p}{dt^2} + p(t)\frac{dy_p}{dt} + q(t)y_p &= u_1{}'y_1{}' + u_1y_1{}'' + u_2{}'y_2{}' + u_2y_2{}'' + p(t)\left(u_1y_1{}' + u_2y_2{}'\right) + q(t)\left(u_1y_1 + u_2y_2\right)\\ &= y_1{}'u_1{}' + y_2{}'u_2{}' = f(t).\end{aligned} \tag{4.15}$$

Observe that (4.14) and (4.15) form a system of two linear equations in the unknowns $u_1{}'$ and $u_2{}'$:

$$\begin{aligned} y_1u_1{}' + y_2u_2{}' &= 0\\ y_1{}'u_1{}' + y_2{}'u_2{}' &= f(t).\end{aligned} \tag{4.16}$$

Applying Cramer's rule gives us

$$u_1{}' = \frac{\begin{vmatrix} 0 & y_2 \\ f(t) & y_2{}' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}} = -\frac{y_2(t)f(t)}{W(S)} \quad \text{and} \quad u_2{}' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1{}' & f(t) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}} = \frac{y_1(t)f(t)}{W(S)}, \tag{4.17}$$

where $W(S)$ is the Wronskian, $W(S) = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}$. After integrating to obtain u_1 and u_2, we form y_p and then a general solution, $y = y_h + y_p$.

Example 107. Solve $y'' + 9y = \sec 3t$, $y(0) = 0$, $y'(0) = 0$, $0 \le t < \pi/6$.

Solution. The corresponding homogeneous equation is $y'' + 9y = 0$ with general solution $y_h = c_1 \cos 3t + c_2 \sin 3t$. Then, a fundamental set of solutions is $S = \{\cos 3t, \sin 3t\}$ and $W(S) = 3$, as we see using `Det`, and `Simplify`.

```
In[652]:= fs = {Cos[3t], Sin[3t]};
          wm = {fs, D[fs, t]};
          wm//MatrixForm

          wd = Simplify[Det[wm]]
```

```
Out[652]= ( Cos[3 t]       Sin[3 t]  )
          ( -3 Sin[3 t]  3 Cos[3 t] )
Out[652]= 3
```

We use (4.17) to find $u_1 = \frac{1}{9} \ln \cos 3t$ and $u_2 = \frac{1}{3}t$.

```
In[653]:= u1 = Integrate[-Sin[3t] Sec[3t]/3, t]

          u2 = Integrate[Cos[3t] Sec[3t]/3, t]
Out[653]= 1/9 Log[Cos[3 t]]
Out[653]= t/3
```

It follows that a particular solution of the nonhomogeneous equation is $y_p = \frac{1}{9} \cos 3t \, \ln \cos 3t + \frac{1}{3}t \sin 3t$ and a general solution is $y = y_h + y_p = c_1 \cos 3t + c_2 \sin 3t + \frac{1}{9} \cos 3t \, \ln \cos 3t + \frac{1}{3}t \sin 3t$.

```
In[654]:= yp = u1 Cos[3t] + u2 Sin[3t]
Out[654]= 1/9 Cos[3 t] Log[Cos[3 t]] + 1/3 t Sin[3 t]
```

Identical results are obtained using `DSolve`.

```
In[655]:= DSolve[y''[t] + 9y[t] == Sec[3t], y[t], t]
Out[655]= {{y[t] -> C[2] Cos[3 t] + 1/9 Cos[3 t] Log[Cos[3 t]] +
              1/3 t Sin[3 t] - C[1] Sin[3 t]}}
```

Applying the initial conditions gives us $c_1 = c_2 = 0$ so we conclude that the solution to the initial value problem is $y = \frac{1}{9} \cos 3t \, \ln \cos 3t + \frac{1}{3}t \sin 3t$.

```
In[656]:= sol = DSolve[
             {y''[t] + 9y[t] == Sec[3t], y[0] == 0, y'[0] == 0}, y[t], t]
Out[656]= {{y[t] -> 1/9 (Cos[3 t] Log[Cos[3 t]] + 3 t Sin[3 t])}}
```

We graph the solution with `Plot` in Figure 4.16.

```
In[657]:= Plot[Evaluate[y[t]/.sol], {t, 0, π/6}]
```

□

4.3 Higher-Order Linear Equations

4.3.1 Basic Theory

The **standard form of the** ***n*****th-order linear equation** is

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t). \tag{4.18}$$

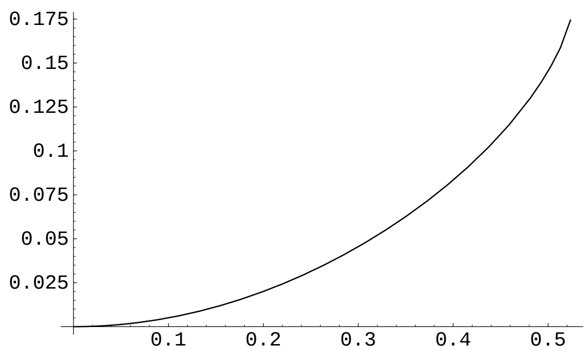


Figure 4.16: The domain of the solution is $-\pi/6 < t < \pi/6$

The **corresponding homogeneous equation** of (4.18) is

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = 0. \tag{4.19}$$

Let $y_1, y_2, \ldots, y_n$ be n solutions of (4.19). The set $S = \left\{y_1, y_2, \ldots, y_n\right\}$ is **linearly independent** if and only if the **Wronskian**,

$$W(S) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ {y_1}' & {y_2}' & {y_3}' & \cdots & {y_n}' \\ {y_1}'' & {y_2}'' & {y_3}'' & \cdots & {y_n}'' \\ {y_1}^{(3)} & {y_2}^{(3)} & {y_3}^{(3)} & \cdots & {y_n}^{(3)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ {y_1}^{(n-1)} & {y_2}^{(n-1)} & {y_3}^{(n-1)} & \cdots & {y_n}^{(n-1)} \end{vmatrix} \tag{4.20}$$

is not identically the zero function. S is **linearly dependent** if S is not linearly independent.

If $y_1, y_2, \ldots, y_n$ are n linearly independent solutions of (4.19), we say that $S = \left\{y_1, y_2, \ldots, y_n\right\}$ is a **fundamental set** for (4.19) and a **general solution** of (4.19) is $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n$.

A **general solution** of (4.18) is $y = y_h + y_p$ where y_h is a general solution of the corresponding homogeneous equation and y_p is a particular solution of (4.18).

4.3.2 Constant Coefficients

If

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = 0$$

has real constant coefficients, we assume that $y = e^{kt}$ and find that k satisfies the **characteristic equation**

$$k^n + a_{n-1}k^{n-1} + \cdots + a_1 k + a_0 = 0. \tag{4.21}$$

If a solution k of (4.21) has multiplicity m, m linearly independent solutions corresponding to k are

$$e^{kt}, te^{kt}, \ldots, t^{m-1}e^{kt}.$$

If a solution $k = \alpha + \beta i$, $\beta \neq 0$, of (4.21) has multiplicity m, $2m$ linearly independent solutions corresponding to $k = \alpha + \beta i$ (and $k = \alpha - \beta i$) are

$$e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t, te^{\alpha t}\cos\beta t, te^{\alpha t}\sin\beta t, \ldots, t^{m-1}e^{\alpha t}\cos\beta t, t^{m-1}e^{\alpha t}\sin\beta t$$

Example 108. Solve $12y''' - 5y'' - 6y' - y = 0$.

Solution. The characteristic equation is

$$12k^3 - 5k^2 - 6k - 1 = (k-1)(3k+1)(4k+1) = 0$$

with solutions $k_1 = -1/3$, $k_2 = -1/4$ and $k_3 = 1$.

`Factor[expression]` attempts to factor *expression*.

```
In[658]:= Factor[12k^3 - 5k^2 - 6k - 1]
Out[658]= (-1 + k) (1 + 3 k) (1 + 4 k)
```

Thus, three linearly independent solutions of the equation are $y_1 = e^{-t/3}$, $y_2 = e^{-t/4}$ and $y_3 = e^t$; a general solution is $y = c_1 e^{-t/3} + c_2 e^{-t/4} + c_3 e^t$. We check with `DSolve`.

```
In[659]:= DSolve[12y'''[t] - 5y''[t] - 6y'[t] - y[t] == 0, y[t], t]
Out[659]= {{y[t] → e^(-t/3) C[1] + e^(-t/4) C[2] + e^t C[3]}}
```

□

Example 109. Solve $y''' + 4y' = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = -1$.

Solution. The characteristic equation is $k^3 + 4k = k(k^2 + 4) = 0$ with solutions $k_1 = 0$ and $k_{2,3} = \pm 2i$ that are found with `Solve`.

Enter `?Solve` to obtain basic help regarding the `Solve` function or see Figure 4.17.

```
In[660]:= Solve[k^3 + 4k == 0]
Out[660]= {{k → 0}, {k → -2 i}, {k → 2 i}}
```

Three linearly independent solutions of the equation are $y_1 = 1$, $y_2 = \cos 2t$, and $y_3 = \sin 2t$. A general solution is $y = c_1 + c_2 \sin 2t + c_3 \cos 2t$.

```
In[661]:= gensol = DSolve[y'''[t] + 4y'[t] == 0, y[t], t]
Out[661]= {{y[t] → C[3] + 1/2 C[1] Cos[2 t] + 1/2 C[2] Sin[2 t]}}
```

Application of the initial conditions shows us that $c_1 = -1/4$, $c_2 = 1/2$, and $c_3 = 1/4$ so the solution to the initial-value problem is $y = -\frac{1}{4} + \frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t$. We verify the computation with `DSolve` and graph the result with `Plot` in Figure 4.18.

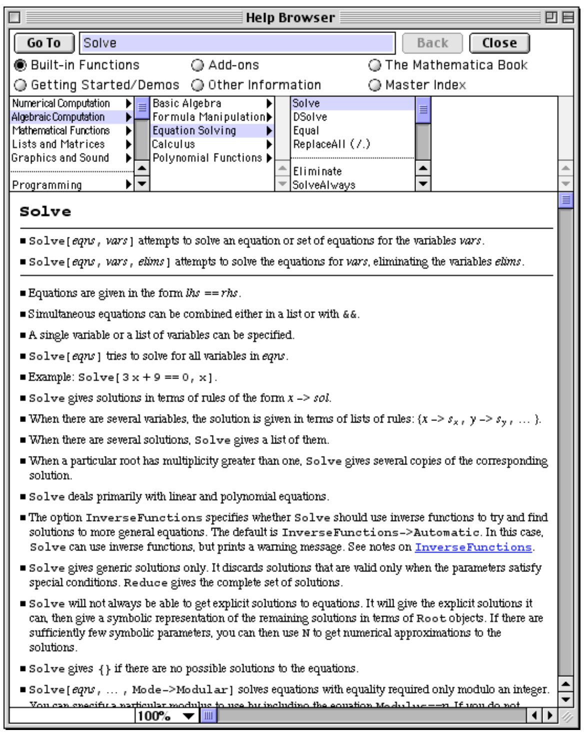


Figure 4.17: If you forget the syntax for a Mathematica function, take advantage of Mathematica's help facility. Here, we use the Help Browser to refresh our memory regarding the `Solve` function.

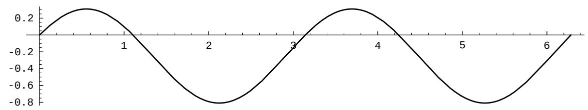


Figure 4.18: Graph of $y = -\frac{1}{4} + \frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t$

```
In[662]:= e1 = y[t]/.gensol[[1]]/.t- > 0
Out[662]= C[1]/2 + C[3]

In[663]:= e2 = D[y[t]/.gensol[[1]], t]/.t- > 0

          e3 = D[y[t]/.gensol[[1]], {t, 2}]/.t- > 0
Out[663]= C[2]
Out[663]= -2 C[1]

In[664]:= cvals = Solve[{e1 == 0, e2 == 1, e3 == -1}]
Out[664]= {{C[1] → 1/2, C[2] → 1, C[3] → -1/4}}

In[665]:= partsol = DSolve[
             {y'''[t] + 4y'[t] == 0, y[0] == 0, y'[0] == 1, y''[0] == -1},
             y[t], t]
Out[665]= {{y[t] → -1/4 + 1/4 Cos[2 t] + 1/2 Sin[2 t]}}

In[666]:= Plot[Evaluate[y[t]/.partsol], {t, 0, 2π},
            AspectRatio- > Automatic]
```

□

Example 110. Find a differential equation with general solution $y = c_1e^{-2t/3}+c_2te^{-2t/3}+c_3t^2e^{-2t/3} + c_4 \cos t + c_5 \sin t + c_6t \cos t + c_7t \sin t + c_8t^2 \cos t + c_9t^2 \sin t$.

Solution. A linear homogeneous differential equation with constant coefficients that has this general solution has fundamental set of solutions

$$S = \left\{e^{-2t/3}, te^{-2t/3}, t^2e^{-2t/3}, \cos t, \sin t, t\cos t, t\sin t, t^2\cos t, t^2\sin t\right\}$$

Hence, in the characteristic equation $k = -2/3$ has multiplicity 3 while $k = \pm i$ has multiplicity 3. The characteristic equation is

$$27\left(k+\frac{2}{3}\right)^3(k-i)^3(k+i)^3 = k^9 + 2k^8 + \frac{13}{3}k^7 + \frac{170}{27}k^6 + 7k^5 + \frac{62}{9}k^4 + 5k^3 + \frac{26}{9}k^2 + \frac{4}{3}k + \frac{8}{27},$$

where we use Mathematica to compute the multiplication with `Expand`.

```
In[667]:= Expand[27(k + 2/3)^3(k^2 + 1)^3]
Out[667]= 8 + 36 k + 78 k^2 + 135 k^3 + 186 k^4 + 189 k^5 + 170 k^6 + 117 k^7 +
            54 k^8 + 27 k^9
```

Thus, a differential equation with the indicated general solution is

$$\frac{d^9y}{dt^9}+2\frac{d^8y}{dt^8}+\frac{13}{3}\frac{d^7y}{dt^7}+\frac{170}{27}\frac{d^6y}{dt^6}+7\frac{d^5y}{dt^5}+\frac{62}{9}\frac{d^4y}{dt^4}+5\frac{d^3y}{dt^3}+\frac{26}{9}\frac{d^2y}{dt^2}+\frac{4}{3}\frac{dy}{dt}+\frac{8}{27}y=0.$$

□

4.3.3 Undetermined Coefficients

For higher-order linear equations with constant coefficients, the method of undetermined coefficients is the same as for second-order equations discussed in Section 4.2.3, provided that the forcing function involves appropriate terms.

Example 111. Solve

$$\frac{d^3y}{dt^3}+\frac{2}{3}\frac{d^2y}{dt^2}+\frac{145}{9}\frac{dy}{dt}=e^{-t},\ y(0)=1,\ \frac{dy}{dt}(0)=2,\ \frac{d^2y}{dt^2}(0)=-1.$$

Solution. The corresponding homogeneous equation, $y'''+\frac{2}{3}y''+\frac{145}{9}y'=0$, has general solution $y_h=c_1+\left(c_2\sin 4t+c_3\cos 4t\right)e^{-t/3}$ and a fundamental set of solutions for the corresponding homogeneous equation is $S=\left\{1,e^{-t/3}\cos 4t,e^{-t/3}\sin 4t\right\}$.

```
In[668]:= DSolve[y'''[t] + 2/3y''[t] + 145/9y'[t] == 0, y[t], t]//
            Simplify
Out[668]= {{y[t] → C[3] + 3/145 e^(-t/3)
              ((12 C[1] - C[2]) Cos[4 t] + (C[1] + 12 C[2]) Sin[4 t])}}
```

For e^{-t}, the associated set of functions is $F=\left\{e^{-t}\right\}$. Because no element of F is an element of S, we assume that $y_p=Ae^{-t}$, where A is a constant to be determined. After defining y_p, we compute the necessary derivatives

```
In[669]:= yp[t_] = a Exp[-t];
          yp'[t]

          yp''[t]

          yp'''[t]
Out[669]= -a e^-t
Out[669]= a e^-t
Out[669]= -a e^-t
```

and substitute into the nonhomogeneous equation.

```
In[670]:= eqn = yp'''[t] + 2/3yp''[t] + 145/9yp'[t] == Exp[-t]
Out[670]= -148/9 a e^-t == e^-t
```

Equating coefficients and solving for A gives us $A=-9/148$ so $y_p=-\frac{9}{148}e^{-t}$ and a general solution is $y=y_h+y_p$.

`SolveAlways[equation,variable]` attempts to solve *equation* so that it is true for all values of *variable*.

```
In[671]:= SolveAlways[eqn, t]
```

Out[671]= $\left\{\left\{a \to -\frac{9}{148}\right\}\right\}$

We verify the result with `DSolve`.

```
In[672]:= gensol = DSolve[y'''[t] + 2/3y''[t] + 145/9y'[t] == Exp[-t],
              y[t], t]
```

Out[672]= $$\left\{\left\{y[t] \to -\frac{9\, e^{-t}}{148} - \left(\frac{3}{145} - \frac{36\, i}{145}\right) e^{\left(-\frac{1}{3}-4\, i\right) t}\, C[1] - \left(\frac{9}{290} - \frac{3\, i}{1160}\right) e^{\left(-\frac{1}{3}+4\, i\right) t}\, C[2] + C[3]\right\}\right\}$$

To obtain a real-valued solution, we use `ComplexExpand`:

```
In[673]:= ?ComplexExpand
"ComplexExpand[expr]expandsexprassuming
   thatallvariablesarereal.ComplexExpand[
   expr, x1, x2, ...]expandsexprassuming
   thatvariablesmatchinganyofthexiarecomplex."
```

```
In[674]:= s1 = ComplexExpand[y[t]/.gensol[[1]]]
```

Out[674]= $$-\frac{9\, e^{-t}}{148} + C[3] - \left(\frac{3}{145} - \frac{36\, i}{145}\right) e^{-t/3}\, C[1]\, \text{Cos}[4\, t] - \left(\frac{9}{290} - \frac{3\, i}{1160}\right) e^{-t/3}\, C[2]\, \text{Cos}[4\, t] + \left(\frac{36}{145} + \frac{3\, i}{145}\right) e^{-t/3}\, C[1]\, \text{Sin}[4\, t] - \left(\frac{3}{1160} + \frac{9\, i}{290}\right) e^{-t/3}\, C[2]\, \text{Sin}[4\, t]$$

```
In[675]:= t1 = Coefficient[s1, Exp[-t/3] Cos[4t]]
```

Out[675]= $\left(-\frac{3}{145} + \frac{36\, i}{145}\right) C[1] - \left(\frac{9}{290} - \frac{3\, i}{1160}\right) C[2]$

```
In[676]:= t2 = Coefficient[s1, Exp[-t/3] Sin[4t]]
```

Out[676]= $\left(\frac{36}{145} + \frac{3\, i}{145}\right) C[1] - \left(\frac{3}{1160} + \frac{9\, i}{290}\right) C[2]$

```
In[677]:= t3 = C[3]
```

Out[677]= $C[3]$

```
In[678]:= Clear[c1, c2, c3]

          s2 = Solve[{t1 == c1, t2 == c2, t3 == c3}, {C[1], C[2], C[3]}]
```

Out[678]= $$\left\{\left\{C[1] \to \left(-\frac{1}{6} - 2\, i\right)(c1 + i\, c2), C[2] \to \left(-16 - \frac{4\, i}{3}\right)(c1 - i\, c2), C[3] \to c3\right\}\right\}$$

The result indicates that the form returned by `DSolve` is equivalent to

```
In[679]:= s3 = s1/.s2[[1]]//Simplify
```

Out[679]= $c3 - \frac{9\, e^{-t}}{148} + c1\, e^{-t/3}\, \text{Cos}[4\, t] + c2\, e^{-t/3}\, \text{Sin}[4\, t]$

To apply the initial conditions, we compute $y(0) = 1$, $y'(0) = 2$ and $y''(0) = -1$

```
In[680]:= e1 = (s3/.t- > 0) == 1

          e2 = (D[s3, t]/.t- > 0) == 2

          e3 = (D[s3, {t, 2}]/.t- > 0) == -1
```

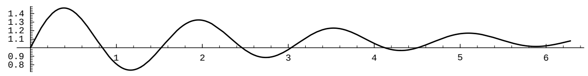


Figure 4.19: The solution of the equation that satisfies $y(0) = 1$, $y'(0) = 2$, and $y''(0) = -1$

Out[680]= $-\frac{9}{148} + c1 + c3 == 1$

Out[680]= $\frac{9}{148} - \frac{c1}{3} + 4\ c2 == 2$

Out[680]= $-\frac{9}{148} - \frac{143\ c1}{9} - \frac{8\ c2}{3} == -1$

and solve for c_1, c_2, and c_3.

```
In[681]:= cvals = Solve[{e1, e2, e3}]
```

Out[681]= $\left\{\left\{c1 \to -\frac{471}{21460}, c2 \to \frac{20729}{42920}, c3 \to \frac{157}{145}\right\}\right\}$

The solution of the initial-value problem is obtained by substituting these values into the general solution.

```
In[682]:= s3/.cvals[[1]]
```

Out[682]= $\frac{157}{145} - \frac{9\ e^{-t}}{148} - \frac{471\ e^{-t/3}\ \mathrm{Cos}[4\ t]}{21460} + \frac{20729\ e^{-t/3}\ \mathrm{Sin}[4\ t]}{42920}$

We check by using `DSolve` to solve the initial-value problem and graph the result with `Plot` in Figure 4.19.

```
In[683]:= sol = DSolve[{y'''[t] + 2/3y''[t] + 145/9y'[t] == Exp[-t],
              y[0] == 1, y'[0] == 2, y''[0] == -1}, y[t],
            t]
```

Out[683]= $\left\{\left\{y[t] \to \frac{157}{145} - \frac{9\ e^{-t}}{148} - \left(\frac{471}{42920} - \frac{20729\ i}{85840}\right)\ e^{\left(-\frac{1}{3}-4\ i\right)\ t} - \left(\frac{471}{42920} + \frac{20729\ i}{85840}\right)\ e^{\left(-\frac{1}{3}+4\ i\right)\ t}\right\}\right\}$

```
In[684]:= realsol = ComplexExpand[y[t]/.sol[[1]]]
```

Out[684]= $\frac{157}{145} - \frac{9\ e^{-t}}{148} - \frac{471\ e^{-t/3}\ \mathrm{Cos}[4\ t]}{21460} + \frac{20729\ e^{-t/3}\ \mathrm{Sin}[4\ t]}{42920}$

```
In[685]:= Plot[realsol, {t, 0, 2π}, AspectRatio- > Automatic]
```

□

Example 112. Solve

$$\frac{d^8y}{dt^8} + \frac{7}{2}\frac{d^7y}{dt^7} + \frac{73}{2}\frac{d^6y}{dt^6} + \frac{229}{2}\frac{d^5y}{dt^5} + \frac{801}{2}\frac{d^4y}{dt^4} + 976\frac{d^3y}{dt^3} + 1168\frac{d^2y}{dt^2} + 640\frac{dy}{dt} + 128y = te^{-t} + \sin 4t + t.$$

Solution. Solving the characteristic equation

```
In[686]:= Solve[k^8 + 7/2k^7 + 73/2k^6 + 229/2k^5+
              801/2k^4 + 976k^3 + 1168k^2 + 640k + 128 ==
            0]
```

```
Out[686]= {{k → -1}, {k → -1}, {k → -1}, {k → -1/2}, {k → -4 i},
           {k → -4 i}, {k → 4 i}, {k → 4 i}}
```

shows us that the solutions are $k_1 = -1/2$, $k_2 = -1$ with multiplicity 3, and $k_{3,4} = \pm 4i$, each with multiplicity 2. A fundamental set of solutions for the corresponding homogeneous equation is

$$S = \left\{e^{-t/2}, e^{-t}, te^{-t}, t^2 e^{-t}, \cos 4t, t\cos 4t, \sin 4t, t \sin 4t\right\}$$

A general solution of the corresponding homogeneous equation is

$$y_h = c_1 e^{-t/2} + \left(c_2 + c_3 t + c_4 t^2\right) e^{-t} + \left(c_5 + c_7 t\right) \sin 4t + \left(c_6 + c_8 t\right) \cos 4t.$$

```
In[687]:= gensol = DSolve[D[y[t], {t, 8}] + 7/2D[y[t], {t, 7}] +
              73/2D[y[t], {t, 6}] + 229/2D[y[t], {t, 5}] +
              801/2D[y[t], {t, 4}] + 976D[y[t], {t, 3}] +
              1168D[y[t], {t, 2}] + 640D[y[t], t] + 128y[t] == 0,
           y[t], t]
Out[687]= {{y[t] →
             e^-t C[1] + e^-t t C[2] + e^-t t^2 C[3] + e^-t/2 C[4] + C[6] Cos[4 t] +
              t C[8] Cos[4 t] - C[5] Sin[4 t] - t C[7] Sin[4 t]}}
```

The associated set of function for te^{-t} is $F_1 = \left\{e^{-t}, te^{-t}\right\}$. We multiply F_1 by t^n, where n is the smallest nonnegative integer so that no element of $t^n F_1$ is an element of S: $t^3 F_1 = \left\{t^3 e^{-t}, t^4 e^{-t}\right\}$. The associated set of functions for $\sin 4t$ is $F_2 = \{\cos 4t, \sin 4t\}$. We multiply F_2 by t^n, where n is the smallest nonnegative integer so that no element of $t^n F_2$ is an element of S: $t^2 F_2 = \left\{t^2 \cos 4t, t^2 \sin 4t\right\}$. The associated set of functions for t is $F_3 = \{1, t\}$. No element of F_3 is an element of S.

Thus, we search for a particular solution of the form

$$y_p = A_1 t^3 e^{-t} + A_2 t^4 e^{-t} + A_3 t^2 \cos 4t + A_4 t^2 \sin 4t + A_5 + A_6 t,$$

where the A_i are constants to be determined.

After defining y_p, we compute the necessary derivatives

We have used `Table` twice for typesetting purposes. You can compute the derivatives using `Table[{n,D[yp[t],{t,n}]},{n,1,8}]`.

```
In[688]:= yp[t_] = a[1]t^3 Exp[-t] + a[2]t^4 Exp[-t] +
           a[3]t^2 Cos[4t] + a[4]t^2 Sin[4t] + a[5] + a[6]t
Out[688]= e^-t t^3 a[1] + e^-t t^4 a[2] + a[5] + t a[6] + t^2 a[3] Cos[4 t] +
           t^2 a[4] Sin[4 t]

In[689]:= Table[{n, D[yp[t], {t, n}]}, {n, 1, 4}]
```

```
Out[689]= {{1, 3 e^-t t^2 a[1] - e^-t t^3 a[1] +
             4 e^-t t^3 a[2] - e^-t t^4 a[2] + a[6] + 2 t a[3] Cos[4 t] +
             4 t^2 a[4] Cos[4 t] - 4 t^2 a[3] Sin[4 t] + 2 t a[4] Sin[4 t]},
          {2, 6 e^-t t a[1] - 6 e^-t t^2 a[1] +
             e^-t t^3 a[1] + 12 e^-t t^2 a[2] - 8 e^-t t^3 a[2] + e^-t t^4 a[2] +
      2 a[3] Cos[4 t] - 16 t^2 a[3] Cos[4 t] + 16 t a[4] Cos[4 t] -
      16 t a[3] Sin[4 t] + 2 a[4] Sin[4 t] - 16 t^2 a[4] Sin[4 t]},
   {3, 6 e^-t a[1] - 18 e^-t t a[1] + 9 e^-t t^2 a[1] - e^-t t^3 a[1] +
      24 e^-t t a[2] - 36 e^-t t^2 a[2] + 12 e^-t t^3 a[2] - e^-t t^4 a[2] -
      96 t a[3] Cos[4 t] + 24 a[4] Cos[4 t] - 64 t^2 a[4] Cos[4 t] -
      24 a[3] Sin[4 t] + 64 t^2 a[3] Sin[4 t] - 96 t a[4] Sin[4 t]},
   {4, -24 e^-t a[1] + 36 e^-t t a[1] - 12 e^-t t^2 a[1] + e^-t t^3 a[1] +
      24 e^-t a[2] - 96 e^-t t a[2] + 72 e^-t t^2 a[2] - 16 e^-t t^3 a[2] +
      e^-t t^4 a[2] - 192 a[3] Cos[4 t] + 256 t^2 a[3] Cos[4 t] -
      512 t a[4] Cos[4 t] + 512 t a[3] Sin[4 t] - 192 a[4] Sin[4 t] +
      256 t^2 a[4] Sin[4 t]}}
```

```
In[690]:= Table[{n, D[yp[t], {t, n}]}, {n, 5, 8}]
```

```
Out[690]= {{5, 60 e^-t a[1] - 60 e^-t t a[1] + 15 e^-t t^2 a[1] - e^-t t^3 a[1] -
             120 e^-t a[2] + 240 e^-t t a[2] - 120 e^-t t^2 a[2] + 20 e^-t t^3 a[2] -
             e^-t t^4 a[2] + 2560 t a[3] Cos[4 t] - 1280 a[4] Cos[4 t] +
             1024 t^2 a[4] Cos[4 t] + 1280 a[3] Sin[4 t] -
             1024 t^2 a[3] Sin[4 t] + 2560 t a[4] Sin[4 t]},
          {6, -120 e^-t a[1] + 90 e^-t t a[1] - 18 e^-t t^2 a[1] + e^-t t^3 a[1] +
             360 e^-t a[2] - 480 e^-t t a[2] + 180 e^-t t^2 a[2] - 24 e^-t t^3 a[2] +
             e^-t t^4 a[2] + 7680 a[3] Cos[4 t] - 4096 t^2 a[3] Cos[4 t] +
             12288 t a[4] Cos[4 t] - 12288 t a[3] Sin[4 t] +
             7680 a[4] Sin[4 t] - 4096 t^2 a[4] Sin[4 t]},
          {7, 210 e^-t a[1] - 126 e^-t t a[1] + 21 e^-t t^2 a[1] - e^-t t^3 a[1] -
             840 e^-t a[2] + 840 e^-t t a[2] - 252 e^-t t^2 a[2] + 28 e^-t t^3 a[2] -
             e^-t t^4 a[2] - 57344 t a[3] Cos[4 t] + 43008 a[4] Cos[4 t] -
             16384 t^2 a[4] Cos[4 t] - 43008 a[3] Sin[4 t] +
             16384 t^2 a[3] Sin[4 t] - 57344 t a[4] Sin[4 t]},
          {8, -336 e^-t a[1] +
             168 e^-t t a[1] - 24 e^-t t^2 a[1] + e^-t t^3 a[1] + 1680 e^-t a[2] -
             1344 e^-t t a[2] + 336 e^-t t^2 a[2] - 32 e^-t t^3 a[2] + e^-t t^4 a[2] -
             229376 a[3] Cos[4 t] + 65536 t^2 a[3] Cos[4 t] -
             262144 t a[4] Cos[4 t] + 262144 t a[3] Sin[4 t] -
             229376 a[4] Sin[4 t] + 65536 t^2 a[4] Sin[4 t]}}
```

and substitute into the nonhomogeneous equation, naming the result `eqn`. At this point we can either equate coefficients and solve for A_i or use the fact that `eqn` is true for *all* values of t.

```
In[691]:= eqn = D[yp[t], {t, 8}] + 7/2D[yp[t], {t, 7}]+
              73/2D[yp[t], {t, 6}] + 229/2D[yp[t], {t, 5}]+
              801/2D[yp[t], {t, 4}] + 976D[yp[t], {t, 3}]+
              1168D[yp[t], {t, 2}] + 640D[yp[t], t] + 128yp[t] ==
            t  Exp[-t] + Sin[4t] + t//
          Simplify
```

```
Out[691]= e^-t (-867 a[1] + 7752 a[2] - 3468 t a[2]+
                 128 e^t a[5] + 640 e^t a[6] + 128 e^t t a[6])-
          64 (369 a[3] - 428 a[4]) Cos[4 t]-
          64 (428 a[3] + 369 a[4]) Sin[4 t] ==
         t + e^-t t + Sin[4 t]
```

We substitute in six values of t

```
In[692]:= sysofeqs = Table[eqn/.t-> n//N, {n, 0, 5}]
```

```
Out[692]= {-867. a[1.]+
               7752. a[2.]-
               64. (369. a[3.]-
                    428. a[4.])+
               128. a[5.]+
               640. a[6.] == 0,
            41.8332 (369. a[3.]-
                    428. a[4.])+
               48.4354 (428. a[3.]+
                    369. a[4.])+
               0.367879 (-867. a[1.]+
                    4284. a[2.]+
                    347.94 a[5.]+
                    2087.64 a[6.]) ==
             0.611077,
            9.312 (369. a[3.]-
                    428. a[4.])-
               63.3189 (428. a[3.]+
                    369. a[4.])+
               0.135335 (-867. a[1.]+
                    816. a[2.]+
                    945.799 a[5.]+
                    6620.59 a[6.]) ==
             3.26003,
            -54.0067 (369. a[3.]-
                    428. a[4.])+
               34.3407 (428. a[3.]+
                    369. a[4.])+
               0.0497871 (-867. a[1.]-
                    2652. a[2.]+
                    2570.95 a[5.]+
                    20567.6 a[6.]) ==
             2.61279,
            61.2902 (369. a[3.]-
                    428. a[4.])+
      18.4258 (428. a[3.]+
           369. a[4.])+
      0.0183156 (-867. a[1.]-
           6120. a[2.]+
           6988.56 a[5.]+
           62897.1 a[6.]) ==
    3.78536,
   -26.1173 (369. a[3.]-
           428. a[4.])-
      58.4285 (428. a[3.]+
           369. a[4.])+
      0.00673795
        (-867. a[1.]-
           9588. a[2.]+
           18996.9 a[5.]+
           189969. a[6.]) ==
    5.94663}
```

and then solve for A_i.

```
In[693]:= coeffs =
            Solve[sysofeqs, {a[1.], a[2.], a[3.], a[4.], a[5.], a[6.]}]
Out[693]= {{a[1.] → -0.00257819,
              a[2.] → -0.000288351,
              a[3.] → -0.0000209413,
              a[4.] → -0.0000180545,
              a[5.] → -0.0390625,
              a[6.] → 0.0078125}}
```

y_p is obtained by substituting the values for A_i into y_p and a general solution is $y = y_h + y_p$. DSolve is able to find an exact solution.

```
In[694]:= gensol = DSolve[D[y[t], {t, 8}] + 7/2D[y[t], {t, 7}] +
                73/2D[y[t], {t, 6}] + 229/2D[y[t], {t, 5}] +
                801/2D[y[t], {t, 4}] + 976D[y[t], {t, 3}] +
                1168D[y[t], {t, 2}] + 640D[y[t], t] + 128y[t] ==
              t Exp[-t] + Sin[4t] + t, y[t], t]//
            Simplify
```

$$\text{Out[694]= } \Big\{\Big\{y[t] \to -\frac{5}{128} - \frac{2924806\, e^{-t}}{24137569} + \frac{t}{128} - \frac{86016\, e^{-t}\, t}{1419857} - \frac{1270\, e^{-t}\, t^2}{83521} - \frac{38\, e^{-t}\, t^3}{14739} - \frac{e^{-t}\, t^4}{3468} + e^{-t}\, C[1] + e^{-t}\, t\, C[2] + e^{-t}\, t^2\, C[3] + e^{-t/2}\, C[4] + \left(\frac{9041976373}{199643253056000} - \frac{107\, t^2}{5109520} + C[6] + t\left(-\frac{1568449}{45168156800} + C[8]\right)\right) \text{Cos}[4\, t] + \left(\frac{13794625331}{798573012224000} + \frac{20406\, t}{352876225} - \frac{369\, t^2}{20438080} - C[5] - t\, C[7]\right) \text{Sin}[4\, t]\Big\}\Big\}$$

□

Variation of Parameters

In the same way as with second-order equations, we assume that a particular solution of the nth order linear equation (4.18) has the form $y_p = u_1(t)y_1 + u_2(t)y_2 + \cdots + u_n(t)y_n$, where $S = \{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions to the corresponding homogeneous equation (4.19). With the assumptions

$$\begin{aligned} y_p{}' &= y_1 u_1{}' + y_2 u_2{}' + \cdots + y_n u_n{}' = 0 \\ y_p{}'' &= y_1{}' u_1{}' + y_2{}' u_2{}' + \cdots + y_n{}' u_n{}' = 0 \\ &\vdots \\ y_p{}^{(n-1)} &= y_1{}^{(n-2)} u_1{}' + y_2{}^{(n-2)} u_2{}' + \cdots + y_n{}^{(n-2)} u_n{}' = 0 \end{aligned} \tag{4.22}$$

we obtain the equation

$$y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n' = f(t). \tag{4.23}$$

Equations (4.22) and (4.23) form a system of n linear equations in the unknowns u_1', $u_2', \ldots, u_n'$. Applying Cramer's rule,

$$u_i' = \frac{W_i(S)}{W(S)}, \tag{4.24}$$

where $W(S)$ is given by (4.20) and $W_i(S)$ is the determinant of the matrix obtained by replacing the ith column of

$$\begin{pmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{pmatrix}.$$

Example 113. Solve $y^{(3)} + 4y' = \sec 2t$.

Solution. A general solution of the corresponding homogeneous equation is $y_h = c_1 + c_2 \cos 2t + c_3 \sin 2t$; a fundamental set is $S = \{1, \cos 2t, \sin 2t\}$ with Wronskian $W(S) = 8$.

```
In[695]:= yh = DSolve[y'''[t] + 4y'[t] == 0, y[t], t]

Out[695]= {{y[t] → C[3] + 1/2 C[1] Cos[2 t] + 1/2 C[2] Sin[2 t]}}

In[696]:= s = {1, Cos[2t], Sin[2t]};
          ws = {s, D[s, t], D[s, {t, 2}]};
          MatrixForm[ws]

Out[696]= (1    Cos[2 t]      Sin[2 t]
           0  -2 Sin[2 t]   2 Cos[2 t]
           0  -4 Cos[2 t]  -4 Sin[2 t])

In[697]:= dws = Simplify[Det[ws]]

Out[697]= 8

In[698]:= dws = Det[ws]//Simplify

Out[698]= 8
```

Using variation of parameters to find a particular solution of the nonhomogeneous equation, we let $y_1 = 1$, $y_2 = \cos 2t$, and $y_3 = \sin 2t$ and assume that a particular solution has the form $y_p = u_1y_1 + u_2y_2 + u_3y_3$. Using the variation of parameters formula, we obtain

$$u_1' = \frac{1}{8}\begin{vmatrix} 0 & \cos 2t & \sin 2t \\ 0 & -2\sin 2t & 2\cos 2t \\ \sec 2t & -4\cos 2t & -4\sin 2t \end{vmatrix} = \frac{1}{4}\sec 2t \quad \text{so} \quad u_1 = \frac{1}{8}\ln|\sec 2t + \tan 2t|,$$

$$u_2' = \frac{1}{8}\begin{vmatrix} 1 & 0 & \sin 2t \\ 0 & 0 & 2\cos 2t \\ 0 & \sec 2t & -4\sin 2t \end{vmatrix} = -\frac{1}{4} \quad \text{so} \quad u_2 = -\frac{1}{4}t$$

and

$$u_3' = \frac{1}{8}\begin{vmatrix} 1 & \cos 2t & 0 \\ 0 & -2\sin 2t & 0 \\ 0 & -4\cos 2t & \sec 2t \end{vmatrix} = -\frac{1}{2}\tan 2t \quad \text{so} \quad u_3 = \frac{1}{8}\ln|\cos 2t|,$$

where we use `Det` and `Integrate` to evaluate the determinants and integrals.

```
In[699]:= u1p = 1/8
              Det[{{0, Cos[2t], Sin[2t]}, {0, -2 Sin[2t], 2 Cos[2t]},
                 {Sec[2t], -4 Cos[2t], -4 Sin[2t]}}]//
             Simplify
```

Out[699]= $\frac{1}{4}$ Sec[2 t]

```
In[700]:= Integrate[u1p, t]
```

Out[700]= $-\frac{1}{8}$ Log[Cos[t] - Sin[t]] + $\frac{1}{8}$ Log[Cos[t] + Sin[t]]

```
In[701]:= u2p = Simplify[1/8 Det[{{1, 0, Sin[2t]}, {0, 0, 2 Cos[2t]},
                 {0, Sec[2t], -4 Sin[2t]}}]]
```

Out[701]= $-\frac{1}{4}$

```
In[702]:= Integrate[u2p, t]
```

Out[702]= $-\frac{t}{4}$

```
In[703]:= u3p = Simplify[1/8 Det[{{1, Cos[2t], 0}, {0, -2 Sin[2t], 0},
                 {0, -4 Cos[2t], Sec[2t]}}]]
```

Out[703]= $-\frac{1}{4}$ Tan[2 t]

```
In[704]:= Integrate[u3p, t]
```

Out[704]= $\frac{1}{8}$ Log[Cos[2 t]]

Thus, a particular solution of the nonhomogeneous equation is

$$y_p = \frac{1}{8}\ln|\sec 2t + \tan 2t| - \frac{1}{4}t\cos 2t + \frac{1}{8}\ln|\cos 2t|\sin 2t$$

and a general solution is $y = y_h + y_p$. We verify the calculations using `DSolve` returns an equivalent solution.

```
In[705]:= gensol =
           DSolve[y'''[t] + 4y'[t] == Sec[2t], y[t], t]//Simplify
```

Out[705]= {{y[t] → $\frac{1}{8}$ (8 C[3] - 2 (t - 2 C[1]) Cos[2 t]-
 Log[Cos[t] - Sin[t]] + Log[Cos[t] + Sin[t]]+
 (4 C[2] + Log[Cos[2 t]]) Sin[2 t])}}

□

4.3.4 Nonlinear Higher-Order Equations

Generally, rigorous results regarding nonlinear equations are very difficult to obtain. In some cases, analysis is best carried out numerically and/or graphically. In other situations, rewriting the equation as a system can be of benefit, which is discussed in the next section. (See Example 117.)

4.4 Systems of Equations

4.4.1 Linear Systems

We now consider first-order linear systems of differential equations:

$$\mathbf{X}' = \mathbf{A}(t)\mathbf{X} + \mathbf{F}(t), \tag{4.25}$$

where

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \dots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

Homogeneous Linear Systems

The corresponding homogeneous system of (4.25) is

$$\mathbf{X}' = \mathbf{A}\mathbf{X}. \tag{4.26}$$

In the same way as with the previously discussed linear equations, a **general solution** of (4.25) is $\mathbf{X} = \mathbf{X}_h + \mathbf{X}_p$ where $\mathbf{X}_h$ is a *general solution* of (4.26) and $\mathbf{X}_p$ is a *particular solution* of the nonhomogeneous system (4.25).

A **particular solution** to a system of ordinary differential equations is a set of functions that satisfy the system but do not contain any arbitrary constants. That is, a particular solution to a system is a set of specific functions, *containing no arbitrary constants*, that satisfy the system.

If $\Phi_1, \Phi_2, \dots, \Phi_n$ are n linearly independent solutions of (4.26), a **general solution** of (4.26) is

$$\mathbf{X} = c_1\Phi_1 + c_2\Phi_2 + \cdots + c_n\Phi_n = \begin{pmatrix} \Phi_1 & \Phi_2 & \dots & \Phi_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \Phi\mathbf{C},$$

where

$$\Phi = \begin{pmatrix} \Phi_1 & \Phi_2 & \dots & \Phi_n \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Φ is called a **fundamental matrix** for (4.26). If Φ is a fundamental matrix for (4.26), $\Phi' = \mathbf{A}\Phi$ or $\Phi' - \mathbf{A}\Phi = \mathbf{0}$.

$\mathbf{A}(t)$ constant

Suppose that $\mathbf{A}(t) = \mathbf{A}$ has constant real entries. Let λ be an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{v}$. Then, $\mathbf{v}e^{\lambda t}$ is a solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$.

If $\lambda = \alpha + \beta i, \beta \neq 0$, is an eigenvalue of $\mathbf{A}$ and has corresponding eigenvector $\mathbf{v} = \mathbf{a} + \mathbf{b}i$, two linearly independent solutions of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ are

$$e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t) \qquad \text{and} \qquad e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t). \tag{4.27}$$

Example 114. Solve each of the following systems. (a) $\mathbf{X}' = \begin{pmatrix} -1/2 & -1/3 \\ -1/3 & -1/2 \end{pmatrix}\mathbf{X}$ (b) $\begin{cases} x' = \frac{1}{2}y \\ y' = -\frac{1}{8}x \end{cases}$ (c) $\begin{cases} dx/dt = -\frac{1}{4}x + 2y \\ dy/dt = -8x - \frac{1}{4}y \end{cases}$

Solution. (a) With `Eigensystem`, we see that the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} -1/2 & -1/3 \\ -1/3 & -1/2 \end{pmatrix}$ are $\lambda_1 = -1/6$ and $\lambda_2 = -5/6$ and $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively.

```
In[706]:= capa = {{-1/2, -1/3}, {-1/3, -1/2}};
          Eigensystem[capa]
Out[706]= {{-5/6, -1/6}, {{1, 1}, {-1, 1}}}
```

Then $\mathbf{X}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t/6}$ and $\mathbf{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-5t/6}$ are two linearly independent solutions of the system so a general solution is $\mathbf{X} = \begin{pmatrix} -e^{-t/6} & e^{-5t/6} \\ e^{-t/6} & e^{-5t/6} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$; a fundamental matrix is $\Phi = \begin{pmatrix} -e^{-t/6} & e^{-5t/6} \\ e^{-t/6} & e^{-5t/6} \end{pmatrix}$.

We use `DSolve` to find a general solution of the system by entering

```
In[707]:= gensol = DSolve[{x'[t] == -1/2x[t] - 1/3y[t],
              y'[t] == -1/3x[t] - 1/2y[t]}, {x[t], y[t]},
            t]
Out[707]= {{x[t] → e^(-5 t/6) C[1] - e^(-t/6) C[2], y[t] → e^(-5 t/6) C[1] + e^(-t/6) C[2]}}
```

We graph the direction field with `PlotVectorField`, which is contained in the `PlotField` package located in the **Graphics** directory, in Figure 4.21.

Remark. After you have loaded the `PlotField` package,

```
PlotVectorField[{f[x,y],g[x,y]},{x,a,b},{y,c,d}]
```

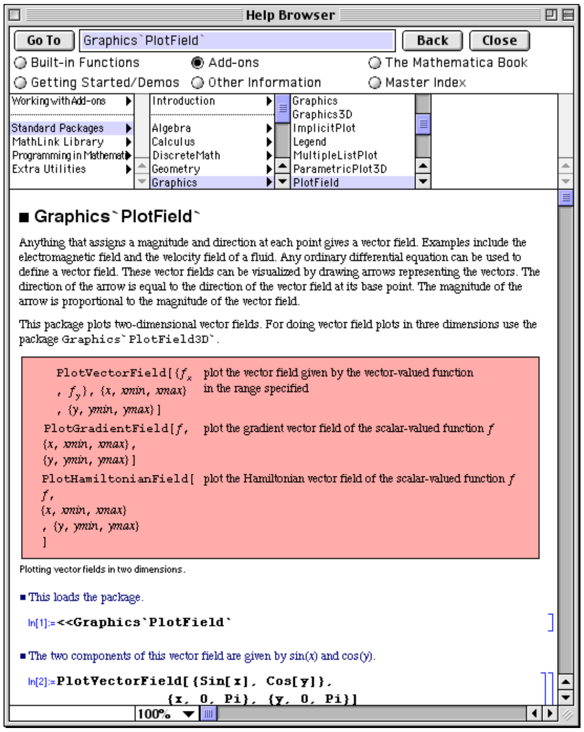


Figure 4.20: The `PlotField` package

generates a basic direction field for the system $\{x' = f(x, y),\ y' = g(x, y)\}$ for $a \le x \le b$ and $c \le y \le d$. (See Figure 4.20.)

```
In[708]:= << Graphics'PlotField'
```

```
In[709]:= pvf = PlotVectorField[{-1/2x - 1/3y, -1/3x - 1/2y},
            {x, -1, 1}, {y, -1, 1}, Axes- > Automatic]
```

Several solutions are also graphed with `ParametricPlot` and shown together with the direction field in Figure 4.22.

```
In[710]:= initsol = DSolve[{x'[t] == -1/2x[t] - 1/3y[t],
              y'[t] == -1/3x[t] - 1/2y[t], x[0] == x0, y[0] == y0},
            {x[t], y[t]}, t]
```

Out[710]= $\left\{\left\{x[t] \to -e^{-5t/6}\left(\frac{1}{2}(-x0 - y0) + \frac{1}{2}e^{2t/3}(-x0 + y0)\right),\ y[t] \to e^{-5t/6}\left(\frac{1}{2}e^{2t/3}(-x0 + y0) + \frac{x0 + y0}{2}\right)\right\}\right\}$

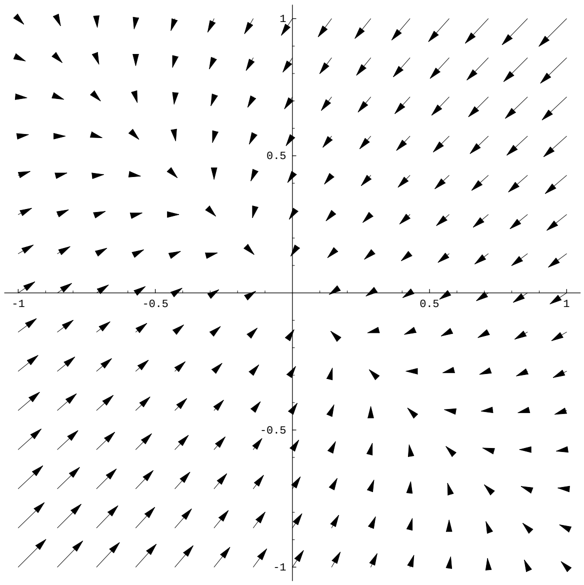


Figure 4.21: Direction field for $\mathbf{X}' = \mathbf{AX}$

```
In[711]:= t1 = Table[ParametricPlot[
              Evaluate[{x[t], y[t]}/.initsol/.{x0- > 1, y0- > i}],
              {t, 0, 15}, DisplayFunction- > Identity,
              PlotStyle- > GrayLevel[0.3]], {i, -1, 1, 2/8}];
 t2 = Table[ParametricPlot[
      Evaluate[{x[t], y[t]}/.initsol/.{x0- > -1, y0- > i}],
      {t, 0, 15}, DisplayFunction- > Identity,
      PlotStyle- > GrayLevel[0.3]], {i, -1, 1, 2/8}];
 t3 = Table[ParametricPlot[
      Evaluate[{x[t], y[t]}/.initsol/.{x0- > i, y0- > 1}],
      {t, 0, 15}, DisplayFunction- > Identity,
      PlotStyle- > GrayLevel[0.3]], {i, -1, 1, 2/8}];
 t4 = Table[ParametricPlot[
      Evaluate[{x[t], y[t]}/.initsol/.{x0- > i, y0- > -1}],
      {t, 0, 15}, DisplayFunction- > Identity,
      PlotStyle- > GrayLevel[0.3]], {i, -1, 1, 2/8}];
```

```
In[712]:= Show[t1, t2, t3, t4,
           pvf, DisplayFunction- > $DisplayFunction,
           AspectRatio- > Automatic]
```

(b) In matrix form the system is equivalent to the system $\mathbf{X}' = \begin{pmatrix} 0 & 1/2 \\ -1/8 & 0 \end{pmatrix}\mathbf{X}$. As in (a), we use `Eigensystem` to see that the eigenvalues and eigenvectors of $mathbf{A} =$

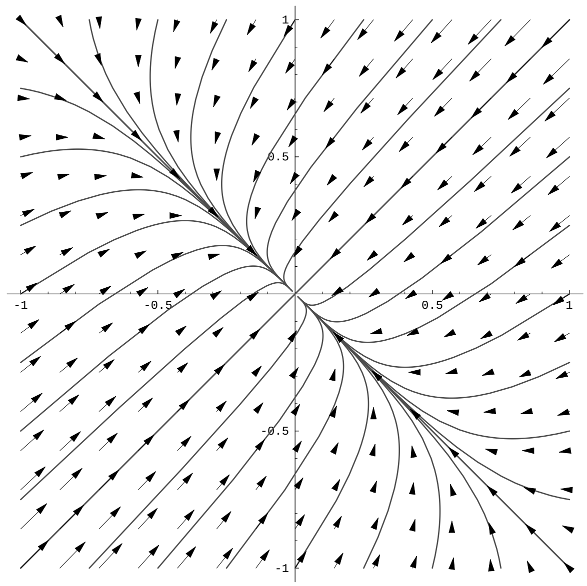


Figure 4.22: Direction field for $\mathbf{X}' = \mathbf{A}\mathbf{X}$ along with various solution curves

$\begin{pmatrix} 0 & 1/2 \\ -1/8 & 0 \end{pmatrix}$ are $\lambda_{1,2} = 0 \pm \frac{1}{4}i$ and $\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} i$.

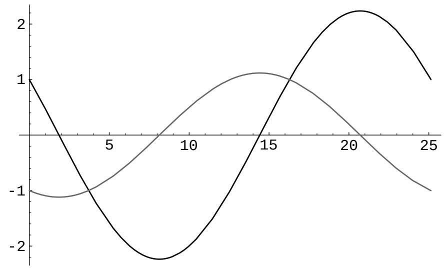
```
In[713]:= capa = {{0, 1/2}, {-1/8, 0}};
          Eigensystem[capa]
Out[713]= {{-i/4, i/4}, {{2 i, 1}, {-2 i, 1}}}
```

Two linearly independent solutions are then $\mathbf{X}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos \frac{1}{4}t - \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \sin \frac{1}{4}t = \begin{pmatrix} \cos \frac{1}{4}t \\ -\frac{1}{2} \sin \frac{1}{4}t \end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin \frac{1}{4}t + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \cos \frac{1}{4}t = \begin{pmatrix} \sin \frac{1}{4}t \\ \frac{1}{2} \cos \frac{1}{4}t \end{pmatrix}$ and a general solution is $\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 = \begin{pmatrix} \cos \frac{1}{4}t & \sin \frac{1}{4}t \\ -\frac{1}{2} \sin \frac{1}{4}t & \frac{1}{2} \cos \frac{1}{4}t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ or $x = c_1 \cos \frac{1}{4}t + c_2 \sin \frac{1}{4}t$ and $y = -c_1 \frac{1}{2} \sin \frac{1}{4}t + \frac{1}{2} c_2 \cos \frac{1}{4}t$.

As before, we use `DSolve` to find a general solution.

```
In[714]:= gensol = DSolve[{x'[t] == 1/2y[t], y'[t] == -1/8x[t]},
            {x[t], y[t]}, t]
Out[714]= {{x[t] -> -2 C[1] Cos[t/4] + 2 C[2] Sin[t/4],
            y[t] -> C[2] Cos[t/4] + C[1] Sin[t/4]}}
```

Initial-value problems for systems are solved in the same way as for other equations. For example, entering



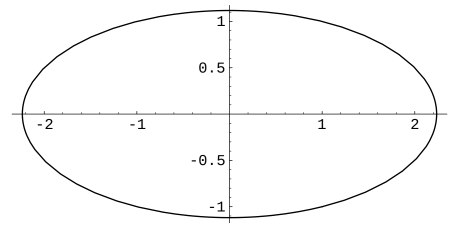


Figure 4.23: (a) Graph of $x(t)$ and $y(t)$ (b) Parametric plot of $x(t)$ versus $y(t)$

```
In[715]:= partsol = DSolve[{x'[t] == 1/2y[t],
              y'[t] == -1/8x[t], x[0] == 1, y[0] == -1}, {x[t], y[t]},
             t]
```

$$\text{Out[715]= } \left\{\left\{x[t] \to -2\left(-\frac{1}{2}\text{Cos}\left[\frac{t}{4}\right]+\text{Sin}\left[\frac{t}{4}\right]\right),\; y[t] \to -\text{Cos}\left[\frac{t}{4}\right]-\frac{1}{2}\text{Sin}\left[\frac{t}{4}\right]\right\}\right\}$$

finds the solution that satisfies $x(0) = 1$ and $y(0) = -1$.

We graph $x(t)$ and $y(t)$ together as well as parametrically with `Plot` and `ParametricPlot`, respectively, in Figure 4.23.

```
In[716]:= p1 = Plot[Evaluate[{x[t], y[t]}/.partsol], {t, 0, 8π},
             PlotStyle- > {GrayLevel[0], GrayLevel[0.4]},
             DisplayFunction- > Identity];
          p2 = ParametricPlot[
             Evaluate[{x[t], y[t]}/.partsol], {t, 0, 8π},
             DisplayFunction- > Identity, AspectRatio- > Automatic];
          Show[GraphicsArray[{p1, p2}]]
```

We can also use `PlotVectorField` and `ParametricPlot` to graph the direction field and/or various solutions as we do next in Figure 4.24.

```
In[717]:= pvf = PlotVectorField[{1/2y, -1/8x}, {x, -2, 2},
             {y, -1, 1}, DisplayFunction- > Identity];
```

```
In[718]:= initsol = DSolve[{x'[t] == 1/2y[t],
              y'[t] == -1/8x[t], x[0] == x0, y[0] == y0},
             {x[t], y[t]}, t]
```

$$\text{Out[718]= } \left\{\left\{x[t] \to -2\left(-\frac{1}{2}\,x0\,\text{Cos}\left[\frac{t}{4}\right]-y0\,\text{Sin}\left[\frac{t}{4}\right]\right),\; y[t] \to y0\,\text{Cos}\left[\frac{t}{4}\right]-\frac{1}{2}\,x0\,\text{Sin}\left[\frac{t}{4}\right]\right\}\right\}$$

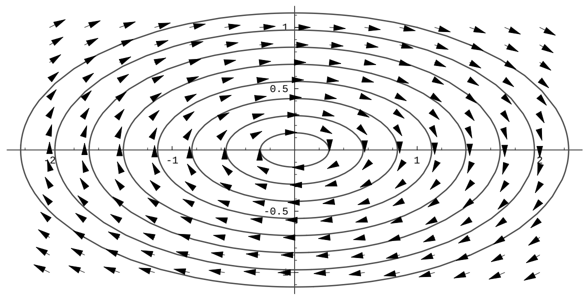


Figure 4.24: Notice that all non-trivial solutions are periodic

```
In[719]:= t1 = Table[ParametricPlot[
              Evaluate[{x[t], y[t]}/.initsol/.{x0- > i, y0- > i}],
              {t, 0, 8π}, DisplayFunction- > Identity,
              PlotStyle- > GrayLevel[0.3]],
            {i, 0, 1, 1/8}];

In[720]:= Show[t1, pvf, DisplayFunction- > $DisplayFunction,
            AspectRatio- > Automatic]
```

(c) In matrix form, the system is equivalent to the system $\mathbf{X}' = \begin{pmatrix} -\frac{1}{4} & 2 \\ -8 & -\frac{1}{4} \end{pmatrix}\mathbf{X}$. The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} -\frac{1}{4} & 2 \\ -8 & -\frac{1}{4} \end{pmatrix}$ are found to be $\lambda_{1,2} = -\frac{1}{4} \pm 4i$ and $\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 2 \end{pmatrix} i$ with `Eigensystem`.

```
In[721]:= capa = {{-1/4, 2}, {-8, -1/4}};
          Eigensystem[capa]
```

Out[721]= $\left\{\left\{-\frac{1}{4}-4\ i, -\frac{1}{4}+4\ i\right\}, \{\{i, 2\}, \{-i, 2\}\}\right\}$

A general solution is then

$$\begin{aligned}
\mathbf{X} &= c_1\mathbf{X}_1 + c_2\mathbf{X}_2 \\
&= c_1 e^{-t/4}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 4t - \begin{pmatrix} 0 \\ 2 \end{pmatrix}\sin 4t\right) + c_2 e^{-t/4}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 4t + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\cos 4t\right) \\
&= e^{-t/4}\left[c_1\begin{pmatrix} \cos 4t \\ -2\sin 4t \end{pmatrix} + c_2\begin{pmatrix} \sin 4t \\ 2\cos 4t \end{pmatrix}\right] = e^{-t/4}\begin{pmatrix} \cos 4t & \sin 4t \\ -2\sin 4t & 2\cos 4t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}
\end{aligned}$$

or $x = e^{-t/4}\left(c_1\cos 4t + c_2\sin 4t\right)$ and $y = e^{-t/4}\left(2c_2\cos 4t - 2c_1\sin 4t\right)$. We confirm this result using `DSolve`.

```
In[722]:= gensol = DSolve[
            {x'[t] == -1/4x[t] + 2y[t], y'[t] == -8x[t] - 1/4y[t]},
            {x[t], y[t]}, t]
```

Out[722]=
$$\{\{x[t]\to C[2]\ \left(-\frac{1}{2}\ i\ \text{Cos}\left[\left(4+\frac{i}{4}\right)\ t\right]+\frac{1}{2}\ i\ \text{Cosh}\left[\left(\frac{1}{4}+4\ i\right)\ t\right]+\frac{1}{2}\ \text{Sin}\left[\left(4+\frac{i}{4}\right)\ t\right]-\frac{1}{2}\ i\ \text{Sinh}\left[\left(\frac{1}{4}+4\ i\right)\ t\right]\right)+C[1]\ \left(-\frac{1}{2}\ \text{Cos}\left[\left(4+\frac{i}{4}\right)\ t\right]-\frac{1}{2}\ \text{Cosh}\left[\left(\frac{1}{4}+4\ i\right)\ t\right]-\frac{1}{2}\ i\ \text{Sin}\left[\left(4+\frac{i}{4}\right)\ t\right]+\frac{1}{2}\ \text{Sinh}\left[\left(\frac{1}{4}+4\ i\right)\ t\right]\right),$$
$$y[t]\to C[2]\ \left(\text{Cos}\left[\left(4+\frac{i}{4}\right)\ t\right]+\text{Cosh}\left[\left(\frac{1}{4}+4\ i\right)\ t\right]+i\ \text{Sin}\left[\left(4+\frac{i}{4}\right)\ t\right]-\text{Sinh}\left[\left(\frac{1}{4}+4\ i\right)\ t\right]\right)+C[1]\ \left(-i\ \text{Cos}\left[\left(4+\frac{i}{4}\right)\ t\right]+i\ \text{Cosh}\left[\left(\frac{1}{4}+4\ i\right)\ t\right]+\text{Sin}\left[\left(4+\frac{i}{4}\right)\ t\right]-i\ \text{Sinh}\left[\left(\frac{1}{4}+4\ i\right)\ t\right]\right)\}\}$$

In[723]:= **`gensol[[1,1,2]]`**

Out[723]=
$$C[2]\ \left(-\frac{1}{2}\ i\ \text{Cos}\left[\left(4+\frac{i}{4}\right)\ t\right]+\frac{1}{2}\ i\ \text{Cosh}\left[\left(\frac{1}{4}+4\ i\right)\ t\right]+\frac{1}{2}\ \text{Sin}\left[\left(4+\frac{i}{4}\right)\ t\right]-\frac{1}{2}\ i\ \text{Sinh}\left[\left(\frac{1}{4}+4\ i\right)\ t\right]\right)+C[1]\ \left(-\frac{1}{2}\ \text{Cos}\left[\left(4+\frac{i}{4}\right)\ t\right]-\frac{1}{2}\ \text{Cosh}\left[\left(\frac{1}{4}+4\ i\right)\ t\right]-\frac{1}{2}\ i\ \text{Sin}\left[\left(4+\frac{i}{4}\right)\ t\right]+\frac{1}{2}\ \text{Sinh}\left[\left(\frac{1}{4}+4\ i\right)\ t\right]\right)$$

In[724]:= **`ComplexExpand[gensol[[1,1,2]]]//Simplify`**

Out[724]=
$$(C[1]\ \text{Cos}[4\ t]-C[2]\ \text{Sin}[4\ t])\ \left(-\text{Cosh}\left[\frac{t}{4}\right]+\text{Sinh}\left[\frac{t}{4}\right]\right)$$

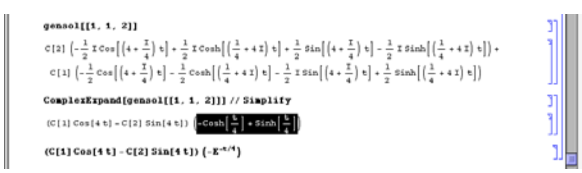
In[725]:= **$(C[1]\ \text{Cos}[4\ t]-C[2]\ \text{Sin}[4\ t])\ \left(-e^{-t/4}\right)$**

In[726]:= **`ComplexExpand[gensol[[1,2,2]]]//Simplify`**

Out[726]=
$$2\ (C[2]\ \text{Cos}[4\ t]+C[1]\ \text{Sin}[4\ t])\ \left(\text{Cosh}\left[\frac{t}{4}\right]-\text{Sinh}\left[\frac{t}{4}\right]\right)$$

In[727]:= **$2\ (C[2]\ \text{Cos}[4\ t]+C[1]\ \text{Sin}[4\ t])\ (e^{-t/4})$**

In this case, we obtained the real form of the solution by selecting the portion of the expression that we wanted to write in terms of exponential functions



and then accessed `TrigToExp` from the **Algebraic Manipulation** palette

to obtain the result.

```
ComplexExpand[gensol[[1, 2, 2]]] // Simplify
2 (C[2] Cos[4 t] + C[1] Sin[4 t]) (Cosh[t/4] - Sinh[t/4])
2 (C[2] Cos[4 t] + C[1] Sin[4 t]) (E^(-t/4))
```

We use `PlotVectorField` and `ParametricPlot` to graph the direction field associated with the system along with various solutions in Figure 4.25.

```
In[728]:= pvf = PlotVectorField[{1/4x + 2y, -8x - 1/4y},
            {x, -1, 1}, {y, -1, 1}, Axes- > Automatic,
            DisplayFunction- > Identity];
```

```
In[729]:= initsol = DSolve[{x'[t] == -1/4x[t] + 2y[t],
              y'[t] == -8x[t] - 1/4y[t], x[0] == x0, y[0] == y0},
            {x[t], y[t]}, t]
```

$$Out[729]= \left\{\left\{x[t] \to x0\ Cos[4\ t]\ Cosh\left[\frac{t}{4}\right] + \frac{1}{2}\ y0\ Cosh\left[\frac{t}{4}\right]\ Sin[4\ t] - x0\ Cos[4\ t]\ Sinh\left[\frac{t}{4}\right] - \frac{1}{2}\ y0\ Sin[4\ t]\ Sinh\left[\frac{t}{4}\right],\right.\right.$$
$$\left.\left. y[t] \to 2\ \left(\frac{1}{2}\ y0\ Cos[4\ t]\ Cosh\left[\frac{t}{4}\right] - x0\ Cosh\left[\frac{t}{4}\right]\ Sin[4\ t] - \frac{1}{2}\ y0\ Cos[4\ t]\ Sinh\left[\frac{t}{4}\right] + x0\ Sin[4\ t]\ Sinh\left[\frac{t}{4}\right]\right)\right\}\right\}$$

```
In[730]:= t1 = Table[ParametricPlot[
              Evaluate[{x[t], y[t]}/.initsol/.{x0- > 1, y0- > i}],
              {t, 0, 15}, DisplayFunction- > Identity,
              PlotStyle- > GrayLevel[0.3]],
            {i, -1, 1, 2/8}];
```

```
In[731]:= Show[t1, pvf, DisplayFunction- > $DisplayFunction,
          PlotRange- > {{-1, 1}, {-1, 1}}, AspectRatio- > Automatic]
```

Last, we illustrate how to solve an initial-value problem and graph the resulting solutions by finding the solution that satisfies the initial conditions $x(0) = 100$ and $y(0) = 10$ and then graphing the results with `Plot` and `ParametricPlot` in Figure 4.26.

```
In[732]:= partsol = DSolve[{x'[t] == -1/4x[t] + 2y[t],
              y'[t] == -8x[t] - 1/4y[t], x[0] == 100, y[0] == 10},
            {x[t], y[t]}, t]
```

$$Out[732]= \left\{\left\{x[t] \to 100\ Cos[4\ t]\ Cosh\left[\frac{t}{4}\right] + 5\ Cosh\left[\frac{t}{4}\right]\ Sin[4\ t] - 100\ Cos[4\ t]\ Sinh\left[\frac{t}{4}\right] - 5\ Sin[4\ t]\ Sinh\left[\frac{t}{4}\right],\right.\right.$$
$$\left.\left. y[t] \to 2\ \left(5\ Cos[4\ t]\ Cosh\left[\frac{t}{4}\right] - 100\ Cosh\left[\frac{t}{4}\right]\ Sin[4\ t] - 5\ Cos[4\ t]\ Sinh\left[\frac{t}{4}\right] + 100\ Sin[4\ t]\ Sinh\left[\frac{t}{4}\right]\right)\right\}\right\}$$

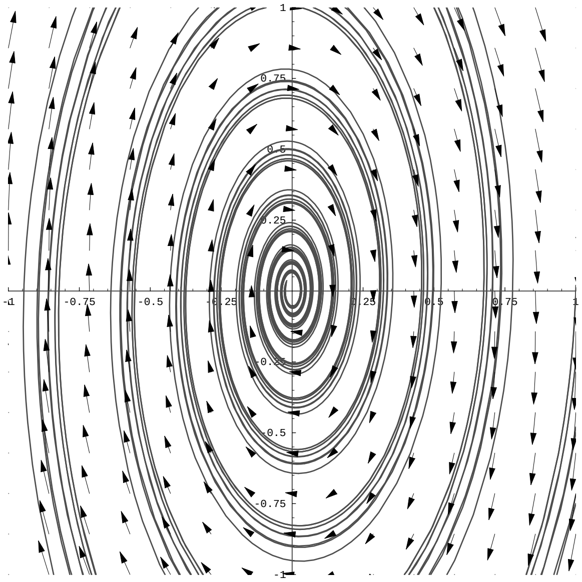


Figure 4.25: Various solutions and direction field associated with the system

```
In[733]:= p1 = Plot[Evaluate[{x[t], y[t]}/.partsol], {t, 0, 20},
              PlotStyle- > {GrayLevel[0], GrayLevel[0.4]},
              DisplayFunction- > Identity, PlotRange- > All];
          p2 = ParametricPlot[
              Evaluate[{x[t], y[t]}/.partsol], {t, 0, 20},
              DisplayFunction- > Identity, AspectRatio- > Automatic];
          Show[GraphicsArray[{p1, p2}]]
```

□

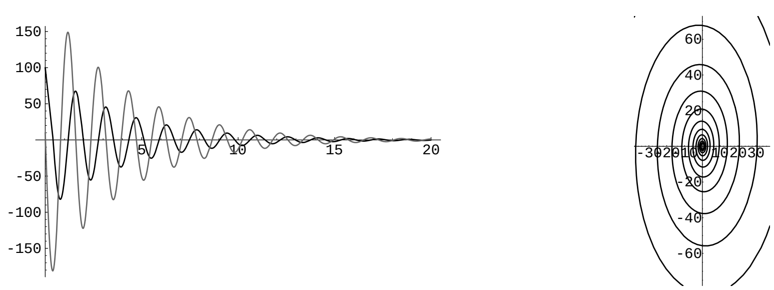


Figure 4.26: (a) Graph of $x(t)$ and $y(t)$ (b) Parametric plot of $x(t)$ versus $y(t)$ (For help with `Show` and `GraphicsArray`, use the Help Browser as shown in Figure 4.27)

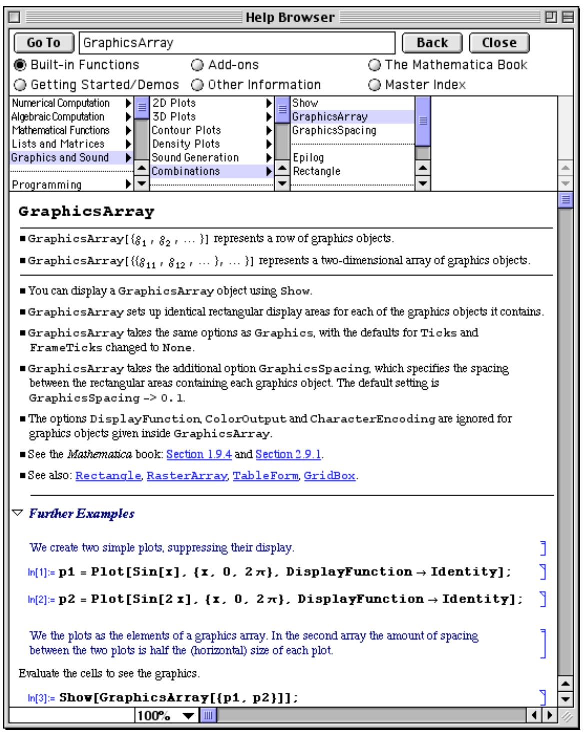


Figure 4.27: Mathematica's help for `GraphicsArray`

4.4.2 Nonhomogeneous Linear Systems

Generally, undetermined coefficients is difficult to implement for nonhomogeneous linear systems as the choice for the particular solution must be very carefully made.

Variation of parameters is implemented in much the same way as for first-order linear equations.

Let $\mathbf{X}_h$ be a general solution to the corresponding homogeneous system of (4.25), $\mathbf{X}$ a general solution of of (4.25), and $\mathbf{X}_p$ a particular solution of (4.25). It then follows that $\mathbf{X} - \mathbf{X}_p$ is a solution to the corresponding homogeneous system so $\mathbf{X} - \mathbf{X}_p = \mathbf{X}_h$ and, consequently, $\mathbf{X} = \mathbf{X}_h + \mathbf{X}_p$.

A particular solution of (4.25) is found in much the same way as with first order linear equations. Let Φ be a fundamental matrix for the corresponding homogeneous system. We assume that a particular solution has the form $\mathbf{X}_p = \Phi\mathbf{U}(t)$. Differentiating $\mathbf{X}_p$ gives us

$$\mathbf{X}_p{}' = \Phi'\mathbf{U} + \Phi\mathbf{U}'.$$

Substituting into (4.25) results in

$$\begin{aligned} \Phi'\mathbf{U} + \Phi\mathbf{U}' &= \mathbf{A}\Phi\mathbf{U} + \mathbf{F} \\ \Phi\mathbf{U}' &= \mathbf{F} \\ \mathbf{U}' &= \Phi^{-1}\mathbf{F} \\ \mathbf{U} &= \int \Phi^{-1}\mathbf{F}\,dt, \end{aligned}$$

where we have used the fact that $\Phi'\mathbf{U} - \mathbf{A}\Phi\mathbf{U} = (\Phi' - \mathbf{A}\Phi)\,\mathbf{U} = \mathbf{0}$. It follows that

$$\mathbf{X}_p = \Phi \int \Phi^{-1}\mathbf{F}\,dt. \tag{4.28}$$

A general solution is then

$$\begin{aligned} \mathbf{X} &= \mathbf{X}_h + \mathbf{X}_p \\ &= \Phi\mathbf{C} + \Phi\int \Phi^{-1}\mathbf{F}\,dt \\ &= \Phi\left(\mathbf{C} + \int \Phi^{-1}\mathbf{F}\,dt\right) = \Phi\int \Phi^{-1}\mathbf{F}\,dt, \end{aligned}$$

where we have incorporated the constant vector $\mathbf{C}$ into the indefinite integral $\int \Phi^{-1}\mathbf{F}\,dt$.

Example 115. Solve the initial-value problem

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 10 & -1 \end{pmatrix}\mathbf{X} - \begin{pmatrix} t\cos 3t \\ t\sin t + t\cos 3t \end{pmatrix}, \quad \mathbf{X}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Remark. In traditional form, the system is equivalent to

$$\begin{cases} x' &= x - y - t\cos 3t \\ y' &= 10x - y - t\sin t - t\cos 3t \end{cases}, \quad x(0) = 1,\ y(0) = -1.$$

Solution. The corresponding homogeneous system is $\mathbf{X}_h' = \begin{pmatrix} 1 & -1 \\ 10 & -1 \end{pmatrix} \mathbf{X}_h$. The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 10 & -1 \end{pmatrix}$ are $\lambda_{1,2} = \pm 3i$ and $\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm \begin{pmatrix} 0 \\ -3 \end{pmatrix} i$, respectively.

```
In[734]:= capa = {{1, -1}, {10, -1}};
          Eigensystem[capa]
Out[734]= {{-3 i, 3 i}, {{1 - 3 i, 10}, {1 + 3 i, 10}}}
```

A fundamental matrix is $\Phi = \begin{pmatrix} \sin 3t & \cos 3t \\ \sin 3t - 3\cos 3t & \cos 3t + 3\sin 3t \end{pmatrix}$ with inverse $\Phi^{-1} = \begin{pmatrix} \frac{1}{3}\cos 3t + \sin 3t & -\frac{1}{3}\cos 3t \\ -\frac{1}{3}\sin 3t + \cos 3t & \frac{1}{3}\sin 3t \end{pmatrix}$.

```
In[735]:= fm = {{Sin[3t], Sin[3t] - 3 Cos[3t]},
              {Cos[3t], Cos[3t] + 3 Sin[3t]}};
          fminv = Inverse[fm]//Simplify
Out[735]= {{1/3 Cos[3 t] + Sin[3 t], Cos[3 t] - 1/3 Sin[3 t]},
            {-1/3 Cos[3 t], 1/3 Sin[3 t]}}
```

We now compute $\Phi^{-1}\mathbf{F}(t)$

```
In[736]:= ft = {-t Cos[3t], -t Sin[t] - t Cos[3t]};
          step1 = fminv.ft
Out[736]= {(-t Cos[3 t] - t Sin[t]) (Cos[3 t] - 1/3 Sin[3 t]) -
             t Cos[3 t] (1/3 Cos[3 t] + Sin[3 t]),
            1/3 t Cos[3 t]^2 + 1/3 (-t Cos[3 t] - t Sin[t]) Sin[3 t]}
```

and $\int \Phi^{-1}\mathbf{F}(t)\,dt$.

```
In[737]:= step2 = Integrate[step1, t]
Out[737]= {1/864 (-288 t^2 + 36 Cos[2 t] -
               216 t Cos[2 t] - 9 Cos[4 t] + 108 t Cos[4 t] - 16 Cos[6 t] +
               48 t Cos[6 t] + 108 Sin[2 t] + 72 t Sin[2 t] - 27 Sin[4 t] -
               36 t Sin[4 t] - 8 Sin[6 t] - 96 t Sin[6 t]), 1/864
            (72 t^2 - 36 Cos[2 t] + 9 Cos[4 t] + 4 Cos[6 t] + 24 t Cos[6 t] -
               72 t Sin[2 t] + 36 t Sin[4 t] - 4 Sin[6 t] + 24 t Sin[6 t])}
```

A general solution of the nonhomogeneous system is then $\Phi\left(\int \Phi^{-1}\mathbf{F}(t)\,dt + \mathbf{C}\right)$.

```
In[738]:= Simplify[fm.step2]
```

```
Out[738]= {1/288 (27 Cos[t] - 4 ((1 + 6 t + 18 t^2) Cos[3 t] + 27 t Sin[t] -
              Sin[3 t] + 6 t Sin[3 t] + 18 t^2 Sin[3 t])),
           1/288 (-36 t Cos[t] - 4 (1 - 6 t + 18 t^2) Cos[3 t] - 45 Sin[t] -
              4 Sin[3 t] - 24 t Sin[3 t] + 72 t^2 Sin[3 t])}
```

It is easiest to use `DSolve` to solve the initial-value problem directly as we do next.

```
In[739]:= check = DSolve[{x'[t] == x[t] - y[t] - t Cos[3t], y'[t] ==
                10x[t] - y[t] - t Sin[t] - t Cos[3t], x[0] == 1, y[0] == -1},
           {x[t], y[t]}, t]
General :: "spell1" : "Possiblespellingerror :
     newsymbolname"check"is similar
      to existing symbol "Check"."
Out[739]= {{x[t] -> 1/288 (-9 Cos[t] + 297 Cos[3 t] - 72 t^2 Cos[3 t] +
              36 t Sin[t] + 192 Sin[3 t] - 24 t Sin[3 t]),
           y[t] ->
             1/288 (-9 Cos[t] - 36 t Cos[t] - 279 Cos[3 t] - 72 t Cos[3 t] -
              72 t^2 Cos[3 t] - 45 Sin[t] + 36 t Sin[t] + 1107 Sin[3 t] -
              24 t Sin[3 t] - 216 t^2 Sin[3 t])}}
```

After using `?Evaluate` to obtain basic information regarding the `Evaluate` function, the solutions are graphed with `Plot` and `ParametricPlot` in Figure 4.28.

```
In[740]:= ?Evaluate
"Evaluate[expr]causesexprtobeevaluatedeven
   ifitappearsastheargumentofafunction
   whoseattributesspecifythatitshouldbe
   heldunevaluated."
In[741]:= p1 = Plot[Evaluate[{x[t], y[t]}/.check], {t, 0, 8π},
              PlotStyle- > {GrayLevel[0], GrayLevel[0.4]},
              DisplayFunction- > Identity];
          p2 = ParametricPlot[
              Evaluate[{x[t], y[t]}/.check], {t, 0, 8π},
              DisplayFunction- > Identity, AspectRatio- > Automatic];
          Show[GraphicsArray[{p1, p2}]]
```

□

4.4.3 Nonlinear Systems

Nonlinear systems of differential equations arise in numerous situations. Rigorous analysis of the behavior of solutions to nonlinear systems is usually very difficult, if not impossible.

To generate numerical solutions of equations, use `NDSolve`. (See Figure 4.9.)

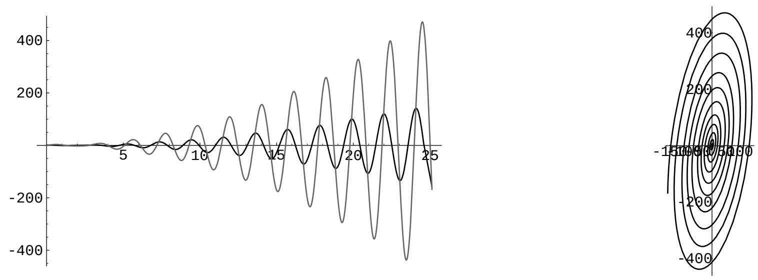


Figure 4.28: (a) Graph of $x(t)$ (in black) and $y(t)$ (in gray) (b) Parametric plot of $x(t)$ versus $y(t)$

Example 116 (Van-der-Pol's equation). Van-der-Pol's equation $x'' + \mu\left(x^2 - 1\right)x' + x = 0$ can be written as the system

$$\begin{aligned} x' &= y \\ y' &= -x - \mu\left(x^2 - 1\right)y. \end{aligned} \tag{4.29}$$

If $\mu = 2/3$, $x(0) = 1$, and $y(0) = 0$, (a) find $x(1)$ and $y(1)$. (b) Graph the solution that satisfies these initial conditions.

Solution. We use `NDSolve` together to solve (4.29) with $\mu = 2/3$ subject to $x(0) = 1$ and $y(0) = 0$. We name the resulting numerical solution `numsol`.

```
In[742]:= numsol = NDSolve[{x′[t] == y[t],
               y′[t] == -x[t] - 2/3(x[t]^2 - 1)y[t], x[0] == 1, y[0] == 0},
             {x[t], y[t]}, {t, 0, 30}]

Out[742]= BoxData({{x[t] → InterpolatingFunction[{{0., 30.}}, " <> "][t], y[t] →
InterpolatingFunction[{{0., 30.}}, " <> "][t]}})
```

We evaluate `numsol` if $t = 1$ to see that $x(1) \approx .5128$ and $y(1) \approx -.9692$.

```
In[743]:= {x[t], y[t]}/.numsol/.t- > 1

Out[743]= {{0.512849, -0.969199}}
```

`Plot`, `ParametricPlot`, and `ParametricPlot3D` are used to graph $x(t)$ and $y(t)$ together in Figure 4.29 (a); a three-dimensional plot, $(t, x(t), y(t))$ is shown in Figure 4.29 (b); a parametric plot is shown in Figure 4.29 (c); and the limit cycle is shown more clearly in Figure 4.29 (d) by graphing the solution for $20 \le t \le 30$.

For help regarding `Show` and `GraphicsArray`, see Figure 4.27.)

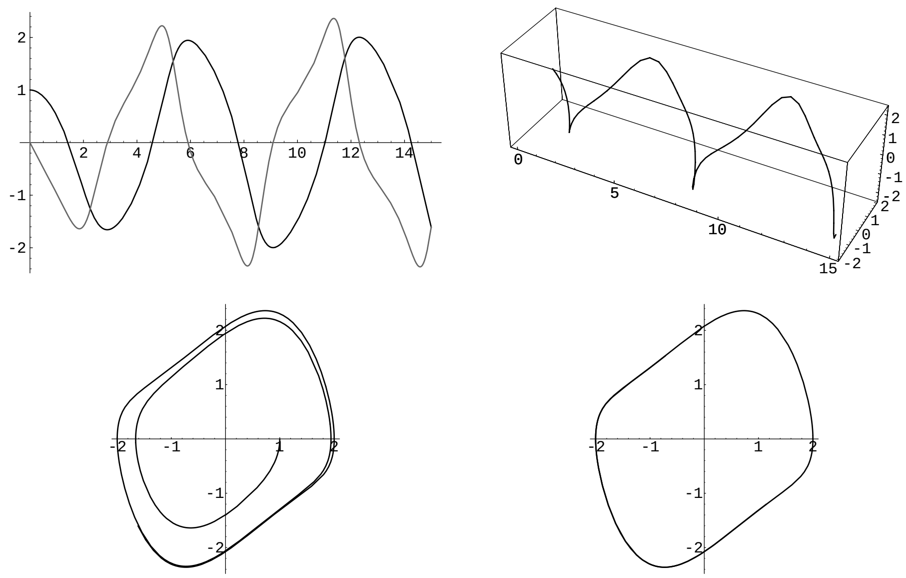


Figure 4.29: (a) $x(t)$ and $y(t)$ (b) A three-dimensional plot (c) $x(t)$ versus $y(t)$ (d) $x(t)$ versus $y(t)$ for $20 \le t \le 30$

```
In[744]:= p1 = Plot[Evaluate[{x[t], y[t]}/.numsol], {t, 0, 15},
              PlotStyle->{GrayLevel[0], GrayLevel[0.4]},
              DisplayFunction->Identity];
          p2 = ParametricPlot3D[Evaluate[{t, x[t], y[t]}/.numsol],
              {t, 0, 15}, DisplayFunction->Identity];
          p3 = ParametricPlot[
              Evaluate[{x[t], y[t]}/.numsol], {t, 0, 15},
              AspectRatio->Automatic, DisplayFunction->Identity];
          p4 = ParametricPlot[
              Evaluate[{x[t], y[t]}/.numsol], {t, 20, 30},
              AspectRatio->Automatic, DisplayFunction->Identity];
          Show[GraphicsArray[{{p1, p2}, {p3, p4}}]]
```

□

Linearization

Consider the autonomous system of the form

An **autonomous system** does not explicitly depend on the independent variable, t. That is, if you write the system omitting all arguments, the independent variable (typically t) does not appear.

$$\begin{aligned} x_1{}' &= f_1\left(x_1, x_2, \ldots, x_n\right) \\ x_2{}' &= f_2\left(x_1, x_2, \ldots, x_n\right) \\ &\vdots \\ x_n{}' &= f_n\left(x_1, x_2, \ldots, x_n\right). \end{aligned} \tag{4.30}$$

An **equilibrium** (or **rest**) **point**, $E = \left(x_1{}^*, x_2{}^*, \ldots, x_n{}^*\right)$, of (4.30) is a solution of the system

$$\begin{aligned} f_1\left(x_1, x_2, \ldots, x_n\right) &= 0 \\ f_2\left(x_1, x_2, \ldots, x_n\right) &= 0 \\ &\vdots \\ f_n\left(x_1, x_2, \ldots, x_n\right) &= 0. \end{aligned} \tag{4.31}$$

The **Jacobian** of (4.30) is

$$\mathbf{J}\left(x_1, x_2, \ldots, x_n\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

The rest point, E, is **locally stable** if and only if all the eigenvalues of $\mathbf{J}(E)$ have negative real part. If E is not locally stable, E is **unstable**.

Example 117 (Duffing's Equation). Consider the forced **pendulum equation** with damping,

$$x'' + kx' + \omega \sin x = F(t). \tag{4.32}$$

Recall the Maclaurin series for $\sin x$: $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$. Using $\sin x \approx x$, (4.32) reduces to the linear equation $x'' + kx' + \omega x = F(t)$.

On the other hand, using the approximation $\sin x \approx x - \frac{1}{6}x^3$, we obtain $x'' + kx' + \omega\left(x - \frac{1}{6}x^3\right) = F(t)$. Adjusting the coefficients of x and x^3 and assuming that $F(t) = F\cos\omega t$ gives us **Duffing's equation**:

$$x'' + kx' + cx + \epsilon x^3 = F\cos\omega t, \tag{4.33}$$

where k and c are positive constants.

Let $y = x'$. Then, $y' = x'' = F\cos\omega t - kx' - cx - \epsilon x^3 = F\cos\omega t - ky - cx - \epsilon x^3$ and we can write (4.33) as the system

$$\begin{aligned} x' &= y \\ y' &= F\cos\omega t - ky - cx - \epsilon x^3 \end{aligned} \tag{4.34}$$

Assuming that $F = 0$ results in the autonomous system

$$\begin{aligned} x' &= y \\ y' &= -cx - \epsilon x^3 - ky. \end{aligned} \tag{4.35}$$

The rest points of system (4.35) are found by solving

$$\begin{aligned} x' &= y \\ y' &= -cx - \epsilon x^3 - ky, \end{aligned}$$

resulting in $E_0 = (0, 0)$.

```
In[745]:= Solve[{y == 0, -c x - ∈ x^3 - k y == 0}, {x, y}]
Out[745]= {{y → 0, x → 0}, {y → 0, x → -(i √c)/√∈}, {y → 0, x → (i √c)/√∈}}
```

We find the Jacobian of (4.35) in `s1`, evaluate the Jacobian at E_0,

```
In[746]:= s1 = {{0, 1}, {-c - 3∈ x^2, -k}};
          s2 = s1/.x- > 0
Out[746]= {{0, 1}, {-c, -k}}
```

and then compute the eigenvalues with `Eigenvalues`.

```
In[747]:= s3 = Eigenvalues[s2]
Out[747]= {1/2 (-k - √(-4 c + k^2)), 1/2 (-k + √(-4 c + k^2))}
```

Because k and c are positive, $k^2 - 4c < k^2$ so the real part of each eigenvalue is always negative if $k^2 - 4c \neq 0$. Thus, E_0 is locally stable.

For the autonomous system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y), \end{aligned}$$

Bendixson's theorem states that if $f_x(x, y) + g_y(x, y)$ is a continuous function that is either always positive or always negative in a particular region R of the plane, then the system has no limit cycles in R. For (4.35) we have

$$\frac{d}{dx}(y) + \frac{d}{dy}\left(-cx - \epsilon x^3 - ky\right) = -k,$$

which is always negative. Hence, (4.35) has no limit cycles and it follows that E_0 is globally, asymptotically stable.

```
In[748]:= D[y, x] + D[-c x - ∈ x^3 - k y, y]
Out[748]= -k
```

We use `PlotVectorField` and `ParametricPlot` to illustrate two situations that occur. In Figure 4.30 (a), we use $c = 1$, $\epsilon = 1/2$, and $k = 3$. In this case, E_0 is a *stable node*. On the other hand, in Figure 4.30 (b), we use $c = 10$, $\epsilon = 1/2$, and $k = 3$. In this case, E_0 is a *stable spiral*.

```
In[749]:= << Graphics`PlotField`

          pvf1 =
            PlotVectorField[{y, -x - 1/2x^3 - 3y}, {x, -2.5, 2.5}, {
              y, -2.5, 2.5}, DisplayFunction- > Identity];

In[750]:= numgraph[init_, c_, opts___] := Module[{numsol},
          numsol = NDSolve[
                {x'[t] == y[t], y'[t] == -c x[t] - 1/2x[t]^3 - 3y[t],
                  x[0] == init[[1]], y[0] == init[[2]]},
                {x[t], y[t]}, {t, 0, 10}];
          ParametricPlot[Evaluate[{x[t], y[t]}/.numsol],
               {t, 0, 10}, opts, DisplayFunction- > Identity]]

In[751]:= i1 = Table[numgraph[{2.5, i}, 1], {i, -2.5, 2.5, 1/2}];
          i2 = Table[numgraph[{-2.5, i}, 1], {i, -2.5, 2.5, 1/2}];
          i3 = Table[numgraph[{i, 2.5}, 1], {i, -2.5, 2.5, 1/2}];
          i4 = Table[numgraph[{i, -2.5}, 1], {i, -2.5, 2.5, 1/2}];

In[752]:= c1 = Show[i1, i2, i3, i4,
              pvf1, PlotRange- > {{-2.5, 2.5}, {-2.5, 2.5}},
              AspectRatio- > Automatic];

In[753]:= pvf2 =
            PlotVectorField[{y, -10x - 1/2x^3 - 3y}, {x, -2.5, 2.5},
              {y, -2.5, 2.5}, DisplayFunction- > Identity];

In[754]:= i1 = Table[numgraph[{2.5, i}, 10], {i, -2.5, 2.5, 1/2}];
          i2 = Table[numgraph[{-2.5, i}, 10], {i, -2.5, 2.5, 1/2}];
          i3 = Table[numgraph[{i, 2.5}, 10], {i, -2.5, 2.5, 1/2}];
          i4 = Table[numgraph[{i, -2.5}, 10], {i, -2.5, 2.5, 1/2}];

In[755]:= c2 = Show[i1, i2, i3, i4,
              pvf2, PlotRange- > {{-2.5, 2.5}, {-2.5, 2.5}},
              AspectRatio- > Automatic];

In[756]:= Show[GraphicsArray[{c1, c2}]]
```

Example 118 (Predator-Prey). The **predator-prey** equations take the form

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = dxy - cy$$

where a, b, c, and d are positive constants. x represents the size of the prey population at time t while y represents the size of the predator population at time t. We use `Solve` to calculate the rest points. In this case, there is one boundary rest point, $E_0 = (0, 0)$ and one interior rest point, $E_1 = (c/d, a/b)$.

```
In[757]:= rps = Solve[{a x - b x y == 0, d x y  - c y == 0}, {x, y}]
Out[757]= {{x → 0, y → 0}, {x → c/d, y → a/b}}
```

The Jacobian is then found using `D`.

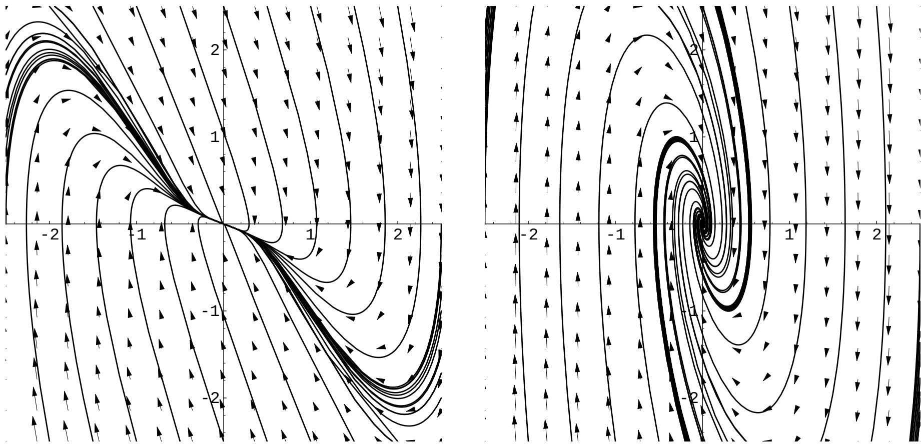


Figure 4.30: (a) The origin is a stable node (b) The origin is a stable spiral

```
In[758]:= jac = {{D[a x - b x y, x], D[a x - b x y, y]},
              {D[d x y  - c y, x], D[d x y  - c y, y]}};
          MatrixForm[jac]
```

Out[758]= $\begin{pmatrix} a - b\,y & -b\,x \\ d\,y & -c + d\,x \end{pmatrix}$

E_0 is unstable because one eigenvalue of $\mathbf{J}(E_0)$ is positive. For the linearized system, E_1 is a center because the eigenvalues of $\mathbf{J}(E_1)$ are complex conjugates.

```
In[759]:= Eigenvalues[jac/.rps[[2]]]
```

Out[759]= $\{-i\sqrt{a}\sqrt{c}, i\sqrt{a}\sqrt{c}\}$

In fact, E_1 is a center for the nonlinear system as illustrated in Figure 4.31, where we have used $a = 1$, $b = 2$, $c = 2$, and $d = 1$. Notice that there are multiple limit cycles around $E_1 = (1/2, 1/2)$.

```
In[760]:= BoxData({<< Graphics‘PlotField‘, pvf = PlotVectorField[{x-2x y, 2x y-
y}, {x, 0, 2}, {y, 0, 2}, DisplayFunction- > Identity]; })

In[761]:= numgraph[init_, opts___] := Module[{numsol},
          numsol = NDSolve[
                 {x′[t] == x[t] - 2x[t]y[t], y′[t] == 2x[t]y[t] - y[t],
                   x[0] == init[[1]], y[0] == init[[2]]},
                 {x[t], y[t]}, {t, 0, 50}];
          ParametricPlot[Evaluate[{x[t], y[t]}/.numsol],
               {t, 0, 10}, opts, DisplayFunction- > Identity]]

In[762]:= i1 = Table[numgraph[{i, i}], {i, 3/20, 1/2, 1/20}];
          Show[i1, pvf, DisplayFunction- > $DisplayFunction,
            PlotRange- > {{0, 2}, {0, 2}}, AspectRatio- > Automatic]
```

In this model, a stable interior rest state is not possible.

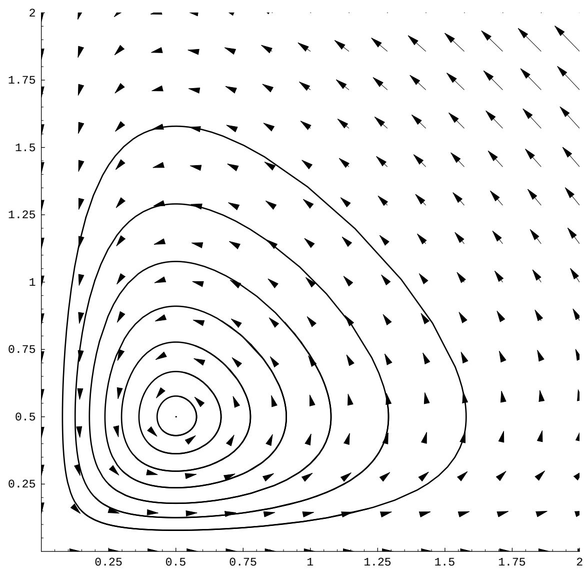


Figure 4.31: Multiple limit cycles about the interior rest point

The complexity of the behavior of solutions to the system increase based on the assumptions made. Typical assumptions include adding satiation terms for the predator (y) and/or limiting the growth of the prey (x). The **standard predator-prey equations of Kolmogorov type**,

$$\begin{aligned} x' &= \alpha x\left(1 - \frac{1}{K}x\right) - \frac{mxy}{a+x} \\ y' &= y\left(\frac{mx}{a+x} - s\right), \end{aligned} \tag{4.36}$$

incorporates both of these assumptions.

We use `Solve` to find the three rest points of system 4.36. Let $E_0 = (0,0)$, $E_1 = (k,0)$ denote the two boundary rest points, and E_2 the interior rest point.

```
In[763]:= rps = Solve[
              {α x (1 - 1/k x) - m x y/(a + x) == 0, y (m x/(a + x) - s) == 0},
              {x, y}]

Out[763]= {{x → 0, y → 0}, {y → 0, x → k},
            {y → - a (-k m + a s + k s) α / (k (m - s)^2), x → - a s / (-m + s)}}
```

The Jacobian, **J**, is calculated next in `s1`.

```
In[764]:= s1 = {{D[α x (1 - 1/k x) - m x y/(a + x), x],
              D[α x (1 - 1/k x) - m x y/(a + x), y]},
            {D[y (m x/(a + x) - s), x], D[y (m x/(a + x) - s), y]}};
          MatrixForm[s1]
```

Out[764]= $\begin{pmatrix} \frac{m\ x\ y}{(a+x)^2} - \frac{m\ y}{a+x} - \frac{x\ \alpha}{k} + \left(1-\frac{x}{k}\right)\ \alpha & -\frac{m\ x}{a+x} \\ \left(-\frac{m\ x}{(a+x)^2}+\frac{m}{a+x}\right)\ y & -s+\frac{m\ x}{a+x} \end{pmatrix}$

Because $\mathbf{J}(E_0)$ has one positive eigenvalue, E_0 is unstable.

```
In[765]:= e0 = s1/.rps[[1]];
          MatrixForm[e0]

          eigs0 = Eigenvalues[e0]
```

Out[765]= $\begin{pmatrix} \alpha & 0 \\ 0 & -s \end{pmatrix}$

Out[765]= $\{-s, \alpha\}$

The stability of E_1 is determined by the sign of $m - s - am/(a + k)$.

```
In[766]:= e1 = s1/.rps[[2]];
          MatrixForm[e1]

          eigs1 = Eigenvalues[e1]
```

Out[766]= $\begin{pmatrix} -\alpha & -\frac{k\ m}{a+k} \\ 0 & \frac{k\ m}{a+k} - s \end{pmatrix}$

Out[766]= $\left\{\frac{k\ m}{a+k} - s, -\alpha\right\}$

The eigenvalues of $\mathbf{J}(E_2)$ are quite complex.

```
In[767]:= e2 = s1/.rps[[3]];
          MatrixForm[e2]

          eigs2 = Eigenvalues[e2]
```

Out[767]= $\begin{pmatrix} \frac{a\ s\ \alpha}{k\ (-m+s)} + \frac{a^2\ m\ s\ (-k\ m+a\ s+k\ s)\ \alpha}{k\ (m-s)^2\ (-m+s)\ \left(a-\frac{a\ s}{-m+s}\right)^2} + \frac{a\ m\ (-k\ m+a\ s+k\ s)\ \alpha}{k\ (m-s)^2\ \left(a-\frac{a\ s}{-m+s}\right)} + \left(1+\frac{a\ s}{k\ (-m+s)}\right)\ \alpha & \frac{a\ m\ s}{(-m+s)\ (a-\cdots} \\ -\frac{a\ (-k\ m+a\ s+k\ s)\ \left(\frac{a\ m\ s}{(-m+s)\ \left(a-\frac{a\ s}{-m+s}\right)^2}+\frac{m}{a-\frac{a\ s}{-m+s}}\right)\ \alpha}{k\ (m-s)^2} & -s-\frac{a\ m\cdots}{(-m+s)\ (\cdots} \end{pmatrix}$

Out[767]= $\left\{\frac{1}{2\ k\ m\ (m-s)}\left(-s\ (a\ m-k\ m+a\ s+k\ s)\ \alpha - \sqrt{\left(-4\ k\ m\ (m-s)\ s\ \left(k\ m^2-a\ m\ s-2\ k\ m\ s+a\ s^2+k\ s^2\right)\ \alpha + s^2\ (a\ m-k\ m+a\ s+k\ s)^2\ \alpha^2\right)}\right), \frac{1}{2\ k\ m\ (m-s)}\left(-s\ (a\ m-k\ m+a\ s+k\ s)\ \alpha + \sqrt{\left(-4\ k\ m\ (m-s)\ s\ \left(k\ m^2-a\ m\ s-2\ k\ m\ s+a\ s^2+k\ s^2\right)\ \alpha + s^2\ (a\ m-k\ m+a\ s+k\ s)^2\ \alpha^2\right)}\right)\right\}$

Instead, we compute the characteristic polynomial of $\mathbf{J}(E_2)$, $p(\lambda) = c_2\lambda^2 + c_1\lambda + c_0$, and examine the coefficients. c_2 is always positive.

```
In[768]:= cpe2 = CharacteristicPolynomial[e2, λ]//Simplify
```

$$\text{Out[768]= } \frac{a\ s\ \alpha\ (m\ (-s+\lambda)+s\ (s+\lambda))+k\ (m-s)\ (-s\ \alpha\ (s+\lambda)+m\ (s\ \alpha+\lambda^2))}{k\ m\ (m-s)}$$

```
In[769]:= c0 = cpe2/.λ-> 0//Simplify
```

$$\text{Out[769]= } \frac{s\ (k\ (m-s)-a\ s)\ \alpha}{k\ m}$$

```
In[770]:= c1 = Coefficient[cpe2, λ]//Simplify
```

$$\text{Out[770]= } \frac{s\ (k\ (-m+s)+a\ (m+s))\ \alpha}{k\ m\ (m-s)}$$

```
In[771]:= c2 = Coefficient[cpe2, λ^2]//Simplify
Out[771]= 1
```

On the other hand, c_0 and $m - s - am/(a + k)$ have the same sign because

```
In[772]:= c0/eigs1[[1]]//Simplify
```

$$\text{Out[772]= } \frac{(a+k)\ s\ \alpha}{k\ m}$$

is always positive. In particular, if $m - s - am/(a + k) < 0$, E_1 is stable. Because c_0 is negative, by Descartes' rule of signs, it follows that $p(\lambda)$ will have one positive root and hence E_2 will be unstable.

On the other hand, if $m - s - am/(a + k) > 0$ so that E_1 is unstable, E_2 may be either stable or unstable. To illustrate these two possibilities let $\alpha = K = m = 1$ and $a = 1/10$. We recalculate.

```
In[773]:= α = 1; k = 1; m = 1; a = 1/10;

In[774]:= rps = Solve[
              {α x (1 - 1/k x) - m x y/(a + x) == 0, y (m x/(a + x) - s) == 0},
              {x, y}]
```

$$\text{Out[774]= } \left\{\{x\to 0, y\to 0\}, \{y\to 0, x\to 1\}, \left\{y\to \frac{10-11\ s}{100\ (-1+s)^2}, x\to -\frac{s}{10\ (-1+s)}\right\}\right\}$$

```
In[775]:= s1 = {{D[α x (1 - 1/k x) - m x y/(a + x), x],
              D[α x (1 - 1/k x) - m x y/(a + x), y]},
             {D[y (m x/(a + x) - s), x], D[y (m x/(a + x) - s), y]}};
           MatrixForm[s1]
```

$$\text{Out[775]= } \begin{pmatrix} 1-2\ x+\frac{x\ y}{\left(\frac{1}{10}+x\right)^2}-\frac{y}{\frac{1}{10}+x} & -\frac{x}{\frac{1}{10}+x} \\ \left(-\frac{x}{\left(\frac{1}{10}+x\right)^2}+\frac{1}{\frac{1}{10}+x}\right)\ y & -s+\frac{x}{\frac{1}{10}+x} \end{pmatrix}$$

```
In[776]:= e2 = s1/.rps[[3]];
           cpe2 = CharacteristicPolynomial[e2, λ]//Simplify
```

$$\text{Out[776]= } \frac{-11\ s^3+s^2\ (21-11\ \lambda)-10\ \lambda^2+s\ (-10+9\ \lambda+10\ \lambda^2)}{10\ (-1+s)}$$

```
In[777]:= c0 = cpe2/.λ-> 0//Simplify
```

$$\text{Out[777]= } s-\frac{11\ s^2}{10}$$

```
In[778]:= c1 = Coefficient[cpe2, λ]//Simplify
Out[778]= (9 - 11 s) s
          ------------
          10 (-1 + s)
In[779]:= c2 = Coefficient[cpe2, λ^2]//Simplify
Out[779]= 1
```

Using `InequalittySolve`, we see that

1. c_0, c_1, and c_2 are positive if $9/11 < s < 10/11$ while
2. c_0 and c_2 are positive and c_1 is negative if $0 < s < 9/11$.

```
In[780]:= << Algebra`InequalitySolve`

          InequalitySolve[c0 > 0 && c1 > 0, s]
Out[780]= 9/11 < s < 10/11
In[781]:= InequalitySolve[c0 > 0 && c1 < 0, s]
Out[781]= 0 < s < 9/11
```

In the first situation, E_2 is stable; in the second E_2 is unstable.

Using $s = 19/22$, we graph the direction field associated with the system as well as various solutions in Figure 4.32. In the plot, notice that all nontrivial solutions approach $E_2 \approx (.63, .27)$; E_2 is stable–a situation that cannot occur with the standard predator-prey equations.

```
In[782]:= rps/.s- > 19/22//N
Out[782]= {{x → 0, y → 0}, {y → 0, x → 1.},
            {y → 0.268889, x → 0.633333}}
In[783]:= << Graphics`PlotField`

          pvf = PlotVectorField[
              α x (1 - 1/k x) - m x y/(a + x) , y (m x/(a + x) - 19/22)}, {
              x, 0, 1}, {y, 0, 1}, DisplayFunction- > Identity];
In[784]:= numgraph[init_, s_, opts___] := Module[{numsol},
          numsol = NDSolve[
                {x'[t] == α x[t] (1 - 1/k x[t]) - m x[t] y[t]/(a + x[t]),
                 y'[t] == y[t] (m x[t]/(a + x[t]) - s),
                 x[0] == init[[1]], y[0] == init[[2]]},
                {x[t], y[t]}, {t, 0, 50}];
          ParametricPlot[Evaluate[{x[t], y[t]}/.numsol],
               {t, 0, 50}, opts, DisplayFunction- > Identity]]
In[785]:= i1 = Table[numgraph[{1, i}, 19/22], {i, 0, 1, 1/10}];
          i2 = Table[numgraph[{i, 1}, 19/22], {i, 0, 1, 1/10}];
          Show[i1, i2,  pvf, DisplayFunction- > $DisplayFunction,
            PlotRange- > {{0, 1}, {0, 1}}, AspectRatio- > Automatic]
```

On the other hand, using $s = 8/11$ (so that E_2 is unstable) in Figure 4.33 we see that all nontrivial solutions appear to approach a limit cycle.

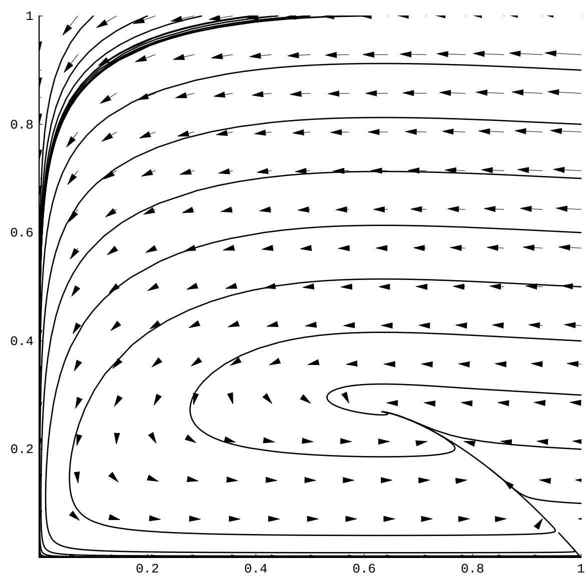


Figure 4.32: $s = 19/22$

```
In[786]:= rps/.s->8/11//N
Out[786]= {{x → 0, y → 0}, {y → 0, x → 1.},
           {y → 0.268889, x → 0.266667}}

In[787]:= i1 = Table[numgraph[{1, i}, 8/11], {i, 0, 1, 1/10}];
          i2 = Table[numgraph[{i, 1}, 8/11], {i, 0, 1, 1/10}];
          p1 = Show[i1, i2,  pvf,
              PlotRange->{{0, 1}, {0, 1}}, AspectRatio->Automatic,
              DisplayFunction->$DisplayFunction]
```

The limit cycle is shown more clearly in Figure 4.34.

```
In[788]:= numgraph[{0.759, 0.262},
            8/11, DisplayFunction->$DisplayFunction,
            PlotRange->{{0, 1}, {0, 1}}, AspectRatio->Automatic]
```

4.5 Exercises

1. Solve each of the following differential equations by hand and verify your result with Mathematica.

   (a) $12y'' + y' - y = 0$

   (b) $9y'' + 6y' + 1 = 0$

   (c) $y'' + \frac{1}{64}y = 0$

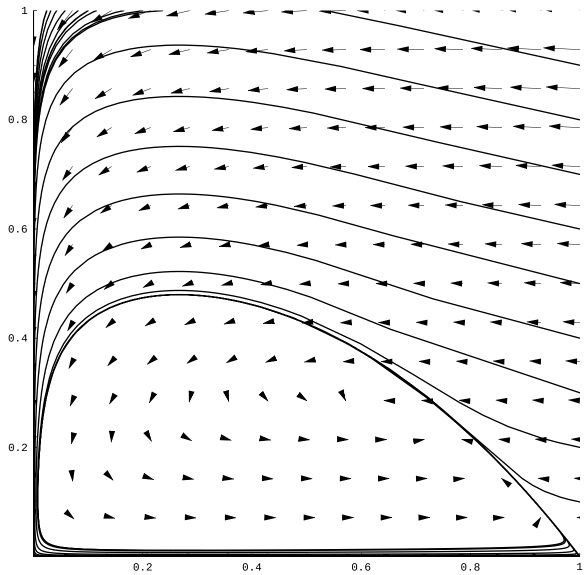


Figure 4.33: $s = 8/11$

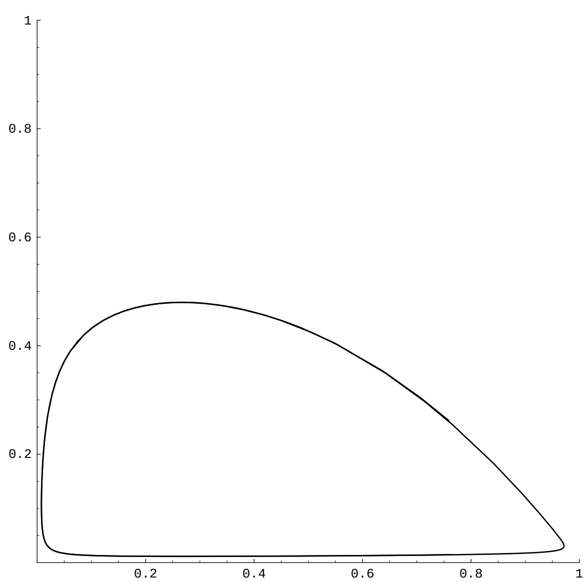


Figure 4.34: A better view of the limit cycle without the direction field

(d) $9y'' + 6y' + 82y = 0$
(e) $y''' - y' = 0$
(f) $y''' - y'' + y' - y = 0$
(g) $y' + ky = 0$, k constant
(h) $y' - y\cot t = \sin t$
(i) $y'' - k^2 y = 0$, $k > 0$ constant
(j) $y'' + k^2 y = 0$, $k > 0$ constant
(k) $y'' + y = 0$, $y(0) = 0$, $y'(0) = 0$
(l) $y'' + y = 1$, $y(0) = 0$, $y'(0) = 0$
(m) $y'' + y = \sin t$, $y(0) = 0$, $y'(0) = 0$
(n) $y'' + y = \sec t$
(o) $y''' - y' = 0$
(p) $y''' + y' = 1$
(q) $y''' + y' = e^t$
(r) $y''' + y' = \sec t$

2. (a) Use Mathematica to solve $y'' - 4y\cot 4t = \sin 4t$. (b) Solve $y'' - 4y\cot 4t = \sin 4t$ by hand and simplify your solution to obtain $y = (C + t)\sin 4t$, where C is an arbitrary constant. (c) Show that the solutions obtained in (a) and (b) are equivalent.

3. Find a differential equation with general solution $y = c_1 e^t + c_2 t e^t + c_3 \cos 2t + c_4 \sin 2t + c_5 t \cos 2t + c_6 t \sin 2t$.

4. Is it possible for a linear differential equation with real constant coefficients to have general solution $y = c_1 t^2 + c_2 t^3$? If so, state a linear differential equation with real constant coefficients that has general solution $y = c_1 t^2 + c_2 t^3$. If not, explain why.

5. Is it possible for a linear differential equation to have general solution $y = c_1 t^2 + c_2 t^3$? If so, state a linear differential equation that has general solution $y = c_1 t^2 + c_2 t^3$. If not, explain why.

6. (a) If there is no forcing (that is, $F = 0$), show that Duffing's equation (see Example 117) can be written as the system $\begin{cases} x' &= y \\ y' &= -cx - \epsilon x^3 - ky \end{cases}$. (b) Find and classify the rest points of this system. (c) Illustrate the stability by graphing various solutions to this system if $k = c = 1$ and $\epsilon = \frac{1}{6}$.

7. (a) Look up the Existence and Uniqueness theorem for first-order linear equations in your text. (b) Does the fact that

$$\frac{dy}{dt} - \frac{1}{t}y = \frac{1}{t^2}(4t\cos 4t - \sin 4t), \qquad y(0) = 0$$

have infinitely many solutions contradict the Existence and Uniqueness theorem? Why or why not?

Bibliography

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Index