Quelques EDP simples résolues avec FreeFem++, Astuces et Trucs

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PLAN

- Historique
- Introduction `Freefem++`
- nouveauté
- Equation de Poisson dépendante du temps
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http://www.freefem.org/

HISTORIQUE

Idée suivre le cours de calcul scientifique, Olivier, Frédéric, M2, UPMC, second semestre

- Macgfem (Olivier , ...)
- freefem (Oliver, Bernardi Dominique, Prud'homme, Frédéric)
- freefem+ (Olivier, Bernardi Dominique, Frédéric)
- freefem++ (Frédéric , Olivier, Antoine Le Hyaric)
- freefem3d (Stephane Del Pino, Olivier)

Comment ça marche, / Dehli, (1992 ?)

c'est mes debuts en C++.

```
typedef double R;
 class Cvirt { public: virtual R operator()(R ) const =0;};

class Cfonc : public Cvirt { public:
  R (*f)(R);                                                    //    la fonction C
  R operator()(R x) const { return (*f)(x);}
  Cfonc( R (*ff)(R)) : f(ff) {} };

class Coper : public Cvirt { public:
  const  Cvirt *g, *d;                                        //    les deux fonctions
  R (*op)(R,R);                                                //    l'opération
  R operator()(R x) const { return (*op)((*g)(x),(*d)(x));}
  Coper( R (*opp)(R,R), const Cvirt *gg, const Cvirt *dd)
    : op(opp),g(gg),d(dd) {}
  ~Coper(){delete g,delete d;} };

static  R Add(R a,R b) {return a+b;}  static  R Sub(R a,R b) {return a-b;}
static  R Mul(R a,R b) {return a*b;}  static  R Div(R a,R b) {return a/b;}
static  R Pow(R a,R b) {return pow(a,b);}
```

The main characteristics of FreeFem++ (2D) I/III

- Problem description (real or complex) by their variational formulations, with access to the internal vectors and matrices if needed.
- Multi-variables, multi-equations, bi-dimensional (or 3D axisymmetric) , static or time dependent, linear or nonlinear coupled systems ; however the user is required to describe the iterative procedures which reduce the problem to a set of linear problems.
- Easy geometric input by analytic description of boundaries by pieces ; however this module is not a CAD system ; for instance when two boundaries intersect, the user must specify the intersection points.
- Automatic mesh generator, based on the Delaunay-Voronoi algorithm. Inner points density is proportional to the density of points on the boundary.

The main characteristics of FreeFem++ (2D) II/III

- Metric-based anisotropic mesh adaptation. The metric can be computed automatically from the Hessian of any FreeFem++ function .
- High level user friendly typed input language with an algebra of analytic and finite element functions.
- Multiple finite element meshes within one application with automatic interpolation of data on different meshes and possible storage of the interpolation matrices.
- A large variety of triangular finite elements : linear and quadratic Lagrangian elements, discontinuous P1 and Raviart-Thomas elements, elements of a non-scalar type, mini-element, ...(no quadrangles).
- Tools to define discontinuous Galerkin formulations via the keywords : "jump" , "mean" , "intalledges").

The main characteristics of FreeFem++ (2D) III/III

- A large variety of linear direct and iterative solvers (LU, Cholesky, Crout, CG, GMRES, UMFPACK) and eigenvalue and eigenvector solvers.
- Near optimal execution speed (compared with compiled C++ implementations programmed directly).
- Online graphics, generation of `,.txt,.eps,.gnu,` `mesh` files for further manipulations of input and output data.
- Many examples and tutorials : elliptic, parabolic and hyperbolic problems, Navier-Stokes flows, elasticity, Fluid structure interactions, Schwarz's domain decomposition method, eigenvalue problem, residual error indicator, ...
- An experimental parallel version using `mpi`

Nouveauté

- Nouveau manuel `freefem++doc.pdf` merci Olivier
- Table de couleur utilisateur
- operateur de C++ `a? b : c` retourne `b` si `a` sinon `b`
- matrice d'interpolation
- algèbre de matrice
- Inéquation variationnelle
- Galerkin discontinue
- couleur dans nedit

Equation de la chaleur

trouver u solution de

$$\partial_t u - \Delta u = f, \quad \text{dans } \Omega \times]0, T[, u(.,0) = u_0, \quad u_{|\Gamma} = g$$

où u_0, f, g, T sont des données.

On l'approche avec un schéma d'Euler implicite :

$u^0 = u_0$

pour $n = 0, N$ avec $\delta t = T/N$ faire

$$u^{n+1} - u^n - \delta t \Delta u^{n+1} = \delta t f, \quad \text{dans } \Omega, u^{n+1}_{|\Gamma} = g$$

La formulation variationnelle : trouver $u^{n+1} \in H^1_0(\Omega)$, $u^{n+1}_{|\Gamma} = g$ et et tel que :

$$\forall v \in H^1_0(\Omega), \quad \int_\Omega (u^{n+1} v + \delta t \nabla u^{n+1} . \nabla v) = \int_\Omega (u^{n+1} v + \delta t f v)$$

Equation de la chaleur / standard

```
 mesh Th=square(100,100) ;
 fespace Vh(Th,P1) ;                                    //    P1 FE space
 Vh uh,vh,u1=0 ;                                  //    unkown and test function.
 func f=1 ;                                        //    right hand side function
 func g=0 ;                                     //    boundary condition function
 real dt =0.01 ;
 int i=0 ;
 problem Poisson(uh,vh,init=i) =                  //    definion of the problem
    int2d(Th)( uh*vh+dt*(dx(uh)*dx(vh) + dy(uh)*dy(vh)) )   //    bil. form
  - int2d(Th)( (u1+dt*f)*vh  )                              //    linear form
  + on(1,2,3,4,uh=g) ;                           //    boundary condition form
...                                                 //    def table de couleur
for (i=0 ;i<10 ;i++)
 {
  Poisson ;                                   //    solve the problem u^{n+1} == uh
  plot(uh,value=true,hsv=colorhsv,viso=viso,fill=1) ;
  u1=uh ;                                                 //    set u^n == u1
}
```

Equation de la chaleur/ table de couleur couleur

```
real[int] colorhsv=[                                    //    color hsv model
  4./6., 1   , 0.5,                                     //    dark blue
  4./6., 1   , 1,                                       //    blue
  5./6., 1   , 1,                                       //    magenta
  1    , 1.  , 1,                                       //    red
  1    , 0.5 , 1                                        //    light red
  ];

  real[int]  viso(20);
  for (int i=0;i<viso.n;i++)
    viso[i]=i*0.005;
```

Equation de la chaleur avec de matrices

```
....

varf  vlaplace(uh,vh) =                              //    definition de problem
  int2d(Th)( uh*vh+ dt*(dx(uh)*dx(vh) + dy(uh)*dy(vh)) )    //    bil. form
  + int2d(Th)( dt*vh*f)  + on(1,2,3,4,uh=g) ;

varf vmasse(u,v) = int2d(Th)(u*v) ;

matrix A = vlaplace(Vh,Vh) ;
set(A,solver=UMFPACK) ;                                   //    factorisation
matrix M = vmasse(Vh,Vh) ;
real [int] b(A.n) ;
real[int] bcl(A.n) ;
bcl = vlaplace(0,Vh) ;                          //    les termes CL + second membre
real[int] in(A.n) ;                  //    un tableau : 1 si interne 0 si frontiere
GetNoBC(A,in) ;                     //    pour pénalisation exact de CL (tgv sur a_ii)
```

Equation de la chaleur avec de matrices

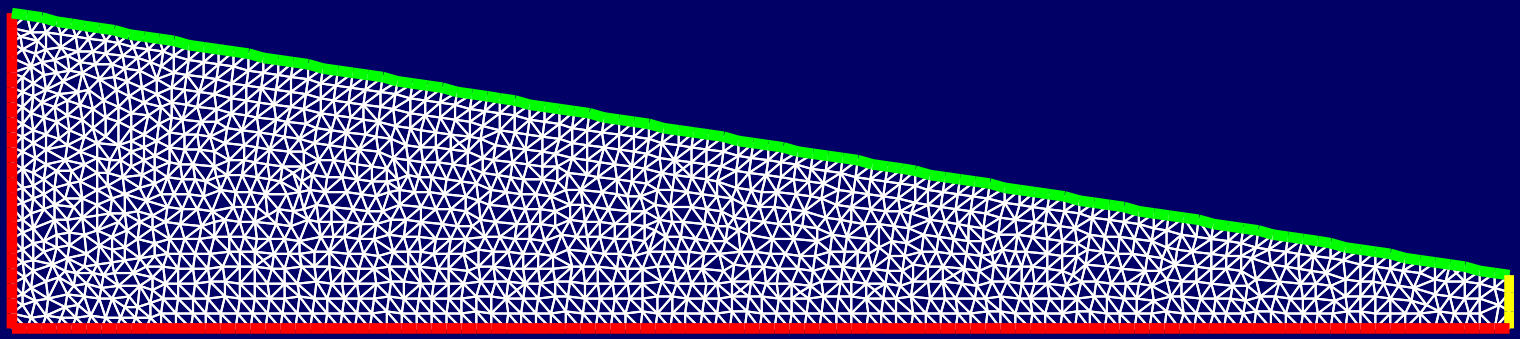
```
for(int i=0;i<10;i++)
  {
    b = M*uh[];
    b = b.* in;                          //    mise a zero des noeud frontiere
    b += bcl;                        //    ajout des terme de CL + second membre
    uh[] = A^-1*b;                                          //    resolution
    plot(uh,value=true);
    cout << i << endl;
  }
```

Equation de la chaleur/ def noeud interne

```
func bool GetNoBC(matrix & A,real[int] & in)
{                                          //    def a array in such what
                                                   //    on unkwnon i
                                       //    in[i] = 1 if no boundary condition
                                          //    in[i] = 0 if boundary condition

  in = A.diag;                              //    take the daig of the matrix
  real tgv = in.max;
  for(int i=0;i<in.n;i++)
    in[i]= in[i] < tgv;                                          //
  return true;
}
```

Un problème de nappe phréatique



The problem is, find p and Ω such that :

$$\begin{cases} -\Delta p = 0 & \text{in } \Omega \\ p = y & \text{on } \Gamma_b \\ \dfrac{\partial p}{\partial n} = 0 & \text{on } \Gamma_d \cup \Gamma_a \\ \dfrac{\partial p}{\partial n} = \frac{q}{K} n_x & \text{on } \Gamma_f \quad (Neumann) \\ p = y & \text{on } \Gamma_f \quad (Dirichlet) \end{cases}$$

where the input water flux is $q = 0.02$, and $K = 0.5$. The velocity u of the water is given by $u = -\nabla p$.

algorithm

We use the following fix point method : let be, $k = 0$, $\Omega^k = \Omega$.

First step, we forgot the Neumann BC and we solve the problem : Find p in $V = H^1(\Omega^k)$, such $p = y$ on Γ_b^k et on Γ_f^k

$$\int_{\Omega^k} \nabla p \nabla p' = 0, \quad \forall p' \in V \text{ with } p' = 0 \text{ on } \Gamma_b^k \cup \Gamma_f^k$$

With the residual of the Neumann boundary condition we build a domain transformation $\mathcal{F}(x, y) = [x, y - v(x)]$ where v is solution of : $v \in V$, such than $v = 0$ on Γ_a^k (bottom)

$$\int_{\Omega^k} \nabla v \nabla v' = \int_{\Gamma_f^k} (\frac{\partial p}{\partial n} - \frac{q}{K} n_x) v', \quad \forall v' \in V \text{ with } v' = 0 \text{ sur } \Gamma_a^k$$

remark : we can use the previous equation to evaluate

$$\int_{\Gamma^k} \frac{\partial p}{\partial n} v' = - \int_{\Omega^k} \nabla p \nabla v'$$

The new domain is : $\Omega^{k+1} = \mathcal{F}(\Omega^k)$

Warning if is the movement is too large we can have triangle overlapping.

```
problem Pp(p,pp,solver=CG) = int2d(Th)( dx(p)*dx(pp)+dy(p)*dy(pp))
  + on(b,f,p=y) ;
problem Pv(v,vv,solver=CG) = int2d(Th)( dx(v)*dx(vv)+dy(v)*dy(vv))
  +  on (a, v=0) + int1d(Th,f)(vv*((Q/K)*N.y)) +  wdpdn[] ) ;
while(errv>1e-6)
{   j++ ;  Pp ;
     wdpdn[] = A*p[] ; wdpdn[] = wdpdn[].*onfree[] ;   wdpdn[] = -wdpdn[] ;
//    hack
    Pv ;    errv=int1d(Th,f)(v*v) ;
     coef = 1 ;
    //    Here french cooking if overlapping see the example
    Th=movemesh(Th,[x,y-coef*v]) ;                          //    deformation
}
```

file:///Users/hecht/Desktop/ffday

Inequation Variationnelle

Le problème est trouver $u \in H_0^1(\Omega)$ tel que

$$u = \underset{u \leq \phi}{argmin}\, J(u), \quad J(u) = \frac{1}{2}\int_\Omega \nabla u.\nabla u - \int_\Omega uf$$

où ϕ est une fonction donnée, et $u = 0$ sur $\Gamma = \partial\Omega$.

Algo :

for $n = ...$

$$-\Delta u^{n+1} = f \quad \text{sur } \Omega\{x/l^n(x) < 0\}$$

où $l^n = u^n - \phi \pm (\Delta u^n + f)$

Inequation Variationnelle

```
mesh Th=square(20,20);
real eps=1e-5;
fespace Vh(Th,P1);                                    //    P1 FE space
int n = Vh.ndof;                                  //    number of Degree of freedom
Vh uh,uhp;                                          //    solution and previous one
Vh Ik;                          //    to def the set where the containt is reached.
real[int] rhs(n);                 //    to store the right and side of the equation
real c=10;                                        //    the parameter of the algoritme
func f=1;                                         //    right hand side function
func fd=0;                                    //    Dirichlet boundary condition function
Vh g=0.05;

                                                  //    array to store
real[int] Aii(n),Aiin(n);                     //    store the diagonal of the matrix
```

```
 real tgv = 1e30 ;      //    a hude value of exact penalisation of boundary
condition
                                  //    the variatonnal form of the problem:
 varf a(uh,vh) =                                  //    definion of the problem
    int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )          //    bilinear form
  - int2d(Th)( f*vh )                                  //    linear form
  + on(1,2,3,4,uh=fd) ;                           //    boundary condition form



                                              //    two version of the problem
matrix A=a(Vh,Vh,tgv=tgv,solver=CG) ;
matrix AA=a(Vh,Vh) ;

                                             //    the mass Matrix construction:
varf vM(uh,vh) = int2d(Th)(uh*vh) ;
matrix M=vM(Vh,Vh) ;          //    to do a fast computing of L^2 norm : sqrt(
u'*(w=M*u))
```

```
Aii=A.diag;                                    //    get the diagonal of the matrix

rhs = a(0,Vh,tgv=tgv);
Ik =0;
uhp=0;
Vh lambda=0;
for(int iter=0;iter<100;++iter)
{
  real[int] b(n); b=rhs;                  //    get a copy of the Right hand side
  real[int] Ak(n);                    //    the complementary of Ik (!Ik = (Ik-1))
                  //    Today the operator Ik- 1. is not implement so we do:
  Ak= 1.; Ak  -= Ik[];                                        //    build Ak =! Ik
   //    adding new locking condition on b and on the diagonal if (Ik ==1 )
  b = Ik[] .* g[];       b *= tgv;      b  -=  Ak .* rhs;
  Aiin = Ik[] *  tgv;       Aiin  +=  Ak  .* Aii;   //    set Aii= tgv i ∈ Ik
  A.diag = Aiin;                                  //    set the matrix diagonal
  set(A,solver=CG);  //    important to change precondiconning for solving
  uh[] = A^-1* b;       //    solve the problem with more locking condition
```

```
  lambda[] = AA * uh[] ;       //    compute the resudal ( fast with matrix)
  lambda[] += rhs ;                                  //    remark rhs = -∫ fv

  Ik = ( lambda + c*( g- uh)) < 0. ;                    //    set the new value

   plot(Ik, wait=1,cmm=" lock set ",value=1 ) ;
   plot(uh,wait=1,cmm="uh") ;
                              //    trick to compute L^2 norm of the variation
      real[int] diff(n),Mdiff(n) ;
      diff= uh[]-uhp[] ;
      Mdiff = M*diff ;
      real err = sqrt(Mdiff'*diff) ;
  cout << "  || u_{k=1} - u_{k} ||_2 " << err << endl ;
  if(err< eps) break ;                                        //    stop test
  uhp[]=uh[] ;                                  //    set the previous solution
}
savemesh(Th,"mm",[x,y,uh*10]) ;
```

Methode de Joint

Methode de joint , Soit $\Omega = \cap_{i=0,..,4}\Omega_i$ une partition sans recouverment. remarque Ω est l'ouvert sans le squelette $\mathcal{S}$ et de frontière externe Γ.

Le probleme avec joint peut s'ecrire, trouver $u \in H^1(\Omega)$ tel que $u_\Gamma = g$ et $\lambda \in L^2(\mathcal{S})$ telle que

$$\forall v \in H^1(\Omega), \quad v_{|\Gamma} = 0, \quad \int_\Omega \nabla u \nabla v + \int_{\mathcal{S}} [v]\lambda = \int_{\Omega_i} fv$$

$$\forall \mu \in L^2(\mathcal{S}), \int_{\mathcal{S}} [u]\mu = 0$$

Donc par sous domaine Ω_i ,

$$\forall v \in H^1(\Omega_i), \quad v_{|\Gamma} = 0, \quad \int_{\Omega_i} \nabla u \nabla v + \int_{\mathcal{S}\cap\partial\Omega_i} \varepsilon_i \lambda v = \int_{\Omega_i} fv$$

et

$$\forall \mu \in L^2(\mathcal{S}), \quad \sum_i \int_{\mathcal{S}\cap\partial\Omega_i} \varepsilon_i \mu u = O$$

où $\varepsilon_i = \mathbf{n}_{\mathcal{S}}.\mathbf{n}_i$ est tels que $\varepsilon_i = \pm 1$ et $\sum_i \varepsilon_i = 0$.

Methode de Joint

```
...                                                    //    def des sous domaine
fespace Lh(Thm,P1) ;
fespace RTh(Tha,[P0edge,P0edge]) ;
 RTh [Nmx,Nmy]=[N.x,N.y] ;                          //    ne marche pas car la normal
                                                   //    n'est definie que sur un bord
varf  vNN([ux,uy],[nx,ny]) = int1d(Tha,1)(( nx*N.x + ny*N.y)/lenEdge) ;
Nmx[]= vNN(0,RTh) ;                                                  //    Def de n_S

plot([Nmx,Nmy],wait=1,cmm="Nmx,Nmy") ;
Lh  lh,rhsl ;
lh=0 ;
rhsl=0 ;
macro AA(u,v)  (dx(u)*dx(v)+dy(u)*dy(v))                                    //
```

Methode de Joint

```
macro defspace(i)
    cout << " Domaine " << i<< " --------" << endl ;
    fespace Vh#i(Th#i,P1) ;      fespace Eh#i(Th#i,P0edge) ;
    Vh#i u#i ;                   Vh#i rhs#i ;
varf veps#i(u,v)= int1d(Th#i,1,qforder=5)( (Nmx*N.x + Nmy*N.y)*v/lenEdge) ;
    Eh#i eps#i = 0 ;
    eps#i[]= veps#i(0,Eh#i) ;
    eps#i = -real(eps#i <-0.01) + real(eps#i >0.01) ;

    varf vLapM#i([u#i],[v#i]) =
        int2d(Th#i)(  AA(u#i,v#i) )    +  int2d(Th#i) (f*v#i)
     +  on(labext,u#i=g)   ;
    varf  vrhsM#i(u#i,v#i) =   on(labext,u#i=g) ;
    varf cc#i([l],[u]) = int1d(Thmm,1)(l*u*eps#i) ;
    matrix C#i = cc#i(Lh,Vh#i) ;
    matrix A#i = vLapM#i(Vh#i,Vh#i,solver=GMRES) ;
    rhs#i[]=vLapM#i(0,Vh#i) ;
                                          //    fin macro defspace(i)
```

Methode de Joint/ def par sous domain

```
func f=1+x+y ;
real g=1 ;

defspace(0)
defspace(1)
defspace(2)
defspace(3)
```

Methode de Joint

```
lh[]=0 ;
varf  vDD(u,v) = int2d(Thm)(u*v*1e-10) ;
verbosity=4 ;
int iter=0 ;
                                                    //

matrix DD=vDD(Lh,Lh) ;
matrix M=[
  [ A0 ,0  ,0  ,0  ,C0 ],
  [ 0  ,A1 ,0  ,0  ,C1 ],
  [ 0  ,0  ,A2 ,0 , C2 ],
  [ 0  ,0  ,0  ,A3, C3 ],
  [ C0',C1',C2',C3',DD ] ] ;

real[int] xx(M.n),  bb(M.n) ;
real[int] bbb=[rhs0[], rhs1[],rhs2[],rhs3[],rhs1[] ] ;
bb=[rhs0[], rhs1[],rhs2[],rhs3[],rhs1[] ] ;
set(M,solver=UMFPACK) ;
xx = M^-1 * bb ;
[u0[],u1[],u2[],u3[],lh[]] = xx ;                   //    dispatcher
```

Conclusion and Future

It is a useful tool to teaches Finite Element Method, and to test some nontrivial algorithm.

– FreeFem by the web (ffw) (P. Havé et A. Le Hyaric)
– No linear technics with automatic differentiation
– 3D
– Suite et FIN. (L'avenir ne manque pas de future et lycée de Versailles)

Thank, for your attention ?